STRONGLY ROBUST ADAPTIVE CONTROL:
THE STRONG ROBUSTNESS APPROACH

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Dit proefschrift is goedgekeurd door :

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en de assistent-promotor, Dr. J.W. Polderman.

To my parents and my sisters Papa, Maman, Anne and Alice

To my husband Gianluca

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Notation

$\mathbb{N}$	the natural numbers
$\mathbb{Z}$	the integers
$\mathbb{R}$	the real numbers
$\mathbb{R}_+$	the positive real numbers
$\mathbb{C}$	the complex numbers
$\mathbb{K}$	$\mathbb{R}$ or $\mathbb{C}$
$\mathbb{K}^{k\times p}$	the $k \times p$ matrices with entries in $\mathbb{K}$
I	the identity matrix with appropriate dimensions
I_p	the identity matrix in $\mathbb{K}^{p\times p}$
$\mathbb{K}[\xi]$	the ring of polynomials in one indeterminate and coefficients in $\mathbb{K}$
$\mathbb{K}^{k\times p}[\xi]$	$k \times p$ polynomials matrices with entries in $\mathbb{K}[\xi]$
$\|a\|$	the 2-norm of the complex number $a \in \mathbb{K}$
$\|\|V\|\|$	the Euclidean norm of the vector $V \in \mathbb{K}^k$
$\|\|M\|\|$	the Frobenius norm of the matrix $M \in \mathbb{K}^{k\times p}$
$\mathcal{P}_n$	the class of single-input/single-output (SISO) systems of order n
$\mathcal{C}_n$	the class of controllable systems in $\mathcal{P}_n$
$\mathcal{S}_n$	the class of asymptotically stable systems in $\mathcal{P}_n$
f	the map assigning to each system $\theta \in \mathcal{C}_n$ its controller
$f(\theta)$	the controller based on the system $\theta \in \mathcal{C}_n$
$f(\Omega)$	the set of controllers based on systems in $\Omega \subset \mathcal{C}_n$
$(\theta, f(\theta'))$	the feedback interconnection of the system $\theta \in \mathcal{C}_n$ and the controller $f(\theta') \in f(\mathcal{C}_n)$
$\chi_{\theta,f(\theta'))}(\xi)$	the characteristic polynomial of $(\theta, f(\theta'))$ for systems $\theta, \theta' \in \mathcal{C}_n$
$r^{\mathbb{K}}(A, D, E)$	the structured stability radius of the Schur matrix $A \in \mathbb{K}^{n\times n}$, with respect to perturbations in $\mathbb{K}^{l\times q}$ with the structure $(D, E) \in \mathbb{K}^{n\times l} \times \mathbb{K}^{q\times n}$
$\rho^{\mathrm{SR}}(\theta)$	the strong robustness radius around a system $\theta \in \mathcal{C}_n$
$\rho^{\mathrm{SQR}}(\theta)$	the strong quadratic robustness radius around a system $\theta \in \mathcal{C}_n$
$\rho^{\mathrm{TISR}}(\theta)$	the time-invariant strong robustness radius around a system $\theta \in \mathcal{C}_n$
$r(\Omega)$	the radius of a ball of systems in $\mathcal{P}_n$
$\sim$	permutation operator on rows and columns of a matrix

Chapter 1

Introduction

The concept of adaptive control, emerged in the mid-fifties, contributed to an immense body of literature and led to many practical applications. So why the need to introduce one more approach? What is the notion of Strong Robustness announced in the title of this thesis and why is it needed for? Our aim in this chapter is to answer these questions so as to motivate the whole thesis, as well as provide an overview of the relevant literature. In Section 1.1, we present the concept of adaptive control, paying particular attention to one of its main paradigms: the certainty equivalence principle. Although this principle is at the origin of most of adaptive control design strategies, it has three well-known disadvantages. After drawing the attention to these three drawbacks, we then speculate about what should be modified in classical adaptive control methods so that the three presented drawbacks vanish. This leads to the notion of Strong Robustness that will be defined and motivated in Section 1.2. Finally, Section 1.3 outlines the structure of the thesis.

1.1 Adaptive control

A very natural start to this thesis would be to define the concept of Adaptive Control. Yet, despite the fifty years of history of this field, one still did not succeed in agreeing on a general definition, mainly because it is not clear how to draw a sharp bound between adaptive control and other control approaches such as robust control. As a first attempt to such a definition, an adaptive control system is viewed in [7] as a control system that *has been designed with an adaptive viewpoint*. Here, *adaptive* characterizes a controller that can modify its behavior in response to 'large' changes in the dynamics of the process to be controlled and the disturbances corrupting this process, where 'large' means that *a single (simple) controller would not be able to cope with such changes* [72]. Alternatively, adaptive control can be seen as the control of a partially unknown system [46], [72], [95]. We adopt this second point of view all along this thesis, leaving the notion of partially unknown system unspecified for the time being.

At first sight, these definitions for adaptive control may leave the impression that an adaptive control system should be able to behave exactly like the non-adaptive control system, obtained when the dynamics of the process to be controlled and the disturbances are not subject to any changes, alternatively when the process to be controlled is completely known. Although achieving such a situation would be ideal, it is clear that the more the process dynamics and the disturbances are corrupted by perturbations, the less we can a priori expect to achieve in terms of control. In other words, the less knowledge we have on this process, the harder it is to control this system. Hence, such ideal goal had to be replaced by a milder one: one ought to construct a control system showing 'reasonably good' performance in spite of uncertainty on the true system. Here, 'reasonably good performance' certainly includes that once the control design is achieved, the performance of the actual controlled system should be close to specified desired performance. However, it is also crucial that *at any time* of the design, the control system is defined and stable, in a sense that will be defined later in this work. Already, many questions arise: how to practically design a controller so that the performance of the unknown system converges to the desired performance, despite initial lack of knowledge? If such a controller exists, beyond the guarantee of a nice asymptotic behavior, can it also ensure that the control system shows an acceptable behavior at any time? If not, what would be the ideal situation providing that at no time of the design bad behaviors are avoided? Each of these three questions are now examined in the following subsections.

1.1.1 The certainty equivalence principle

Intuitively, the better the system to be controlled is known, the better a controller designed on a guess of this plant may be expected to perform when applied to this real plant. Another intuitive idea is that for the previous idea to be true, the uncertainty on the true system must be sufficiently small. As we shall see later in Section 1.1.2: indeed, given two large uncertainty levels, there is no real guarantee that the guess on the system to be controlled obtained for the lower of these two large uncertainty levels will be better than the guess obtained for the higher uncertainty level.
Nevertheless, the idea consisting in relying on a guess of the system to be controlled to perform its control is known as the *Certainty Equivalence Principle* and is the cement of a wide spectrum of classical adaptive control approaches [7], [14], [72]. At each time of the design, based on the available knowledge on the true plant and on a selection law, one constructs an approximation of this true plant, *the model*, also called *estimate*. This constitutes the *identification* step. Then, the identified model is used for on-line controller design without any regards for errors between this model and the true system which generated the data. Further, one applies the model-based controller to the real system and compare the performance of the resulting closed-loop system with the desired control performance. If the performance mismatch is not small enough, one then constructs the new model on the basis of the previous estimate and the new data measurement, subsequently re-tune the model-based controller, until the closed-loop performance is close enough to the desired one. It is clear that the task targeted during such a strategy is to obtain a good controller. Since the controller is based on an estimate of the plant, one way to obtain a controller that becomes good enough is to decrease the model error. At the limit, the model would then converge to the real system parameter and hence the model-based controller would approach the controller designed on the basis of the true system. However, the certainty equivalence controller ignores the plant/model mismatch and adapts its control action so as to meet the control objective for the estimated system. Hence there is nor the guarantee neither a real probing of the system to decrease the uncertainty. This issue, called *the identifiability problem in adaptive control*,

has been extensively studied in the system identification literature [91], [95]: due to lack of internal excitation, the control law applied in closed-loop may make some of the unknown system parameters invisible to the identification process. In turn, the identifiability problem may result in control performances degradation [18], [43], [103], [90]. Ionnaou1 In this respect, a major result is that the input, in order to yield a sequence of models leading to a good control asymptotically, must be sufficiently exciting [43], [72], [18]. In addition, for stochastic systems, it has been proved that in minimum variance adaptive control, despite the fact that the estimate normally converges to a model that is different from the real parameter, the model-based controller asymptotically equals the controller we would obtain when using the exact system parameter [95], [97]. Moreover, it has been established that in the case of adaptive pole assignment, the information one may obtain from the closed-loop behavior of the system is sufficient to generate the proper sequence of control inputs [72],[91]. Hence, in that case, even if the plant parameters are not exactly identified, the generated control law asymptotically equals the control law we would have obtained on the basis of the complete knowledge of the system. In this thesis, the identifiability issue will not be further discussed. We will mainly focus on the case of adaptive pole placement design.

1.1.2 Three issues in certainty equivalence adaptive control

As discussed in the previous subsection, certainty equivalence is key in most adaptive control designs. We saw that under some conditions, which will be assumed to be satisfied throughout this thesis, the use of this paradigm yields a controller that asymptotically generates the proper control input sequence, i.e., the control input sequence we would obtain when designing it on the basis of the true system to be controlled. This is due to the fact that with time, the uncertainty on the true system becomes small enough to yield a model sequence resulting in a good design. To be more specific, asymptotically, the time-varying model is controllable (hence, at least stabilizable). Moreover the fixed controller based on the frozen model at each time stabilizes the true plant asymptotically. However, when little information is available on the real system, as it is common to be in the initial phase of an adaptive control design, it is likely that the model is poor from a control point of view. Hence controllability of the model and stabilizability of the model-based controller may not apply. This may cause severe problems when using classical adaptive control design.

Pole-zero cancellation problem in adaptive control

As it turns out to be often the case in the initial phase of an adaptive control design, we have very little prior knowledge on the level of controllability of the system to be controlled, that is, the distance from this system to the set of uncontrollable systems. Hence, the model provided by the update law that is defined by the adaptive control algorithm might be not controllable. However guaranteeing the controllability of the estimated system is crucial, since otherwise global stability of the adaptive scheme might be completely disrupted. As a matter of fact, to apply many well-established stability and performance results, one has to suppose that the estimated model satisfies a uniform controllability assumption ([31], [99], [101]). On the other hand, classical identification approaches ([35], [64], [72], [87]) do not guarantee such a controllability property in the absence of suitable excitation conditions, which is often the case in closed-loop identification. This issue, known as *the pole-zero*

cancellation problem in adaptive control, attracted significant attention in the literature, in stochastic and deterministic settings. In a stochastic setting, a first way to view this problem relies on the property that the sets of parameters corresponding to non-controllable models is a proper algebraic variety. Hence the event of getting such a model during the finite time of the identification process has probability zero [75], i.e., will never occur in practice. This is to some extent true, however, ignoring the problem is unsatisfactory. Moreover, the probability of getting an uncontrollable model can become positive in the limit and, to the best of our knowledge, no result exists to indicate for which control laws this may occur.
A large body of the literature in adaptive control analysis deals with the pole/zero cancellation problem in adaptive control. Some of the main approaches to face this problem are now listed, without any claim of completeness.

- A first class of approaches consists of the a-posteriori modification of the estimate, e.g., the least squares estimate, so that it stays in or converges to a set of controllable systems ([30], [43], [68], [69], [92], [112]). This can be done by using properties of its covariance matrix ([68], [69], [92]) or by projecting the estimate on a set of controllable systems whilst assuring that the modified estimate inherits some useful properties of the original estimate ([30], [43], [112]). In this manner, before using the estimate for the control design, controllability of the model is secured. However, the main drawback of this approach is its computational complexity which tremendously increases with the order of the system to be controlled [69].

- A second family of approaches amounts in modifying the identification algorithm so as to force the estimate to belong to an a priori known set of controllable models containing the true parameters ([61], [62], [76], [87], [96]). The requirement of the knowledge of a set of controllable systems containing the true unknown system is however a significant limitation of such approaches, confining their use to the cases where the parameter uncertainty is highly structured.

- A third body of approaches ([21], [82], [94]) results in keeping the system estimates away from the uncontrollable models through application of exciting signals amongst three types of signals: persistent, asymptotically vanishing or sporadically appearing when necessary. Again, these approaches become computationally expensive as the system order increases.

- Finally, worth to be mentioned are alternative approaches in which other methods than the classical certainty equivalence strategy are used to design the controller. Such an approach, presented in [89], is a cyclic switching control strategy, steering periodically the unknown system according to a specified logic. Another approach [6] lies in alternative parameterizations of the system to be controlled such that the problem of avoiding non-controllability is avoided or non-existent.

These reported methods offer a clear analysis of the unavoidability of the pole-zero cancellation problem in adaptive control and propose various solutions to face this problem. Their main drawback, however, remain in their computational cost or the assumption that the system to be controlled belongs to a known convex set of controllable systems.

Non-stabilizing model-based controller

If the uncertainty on the system to be controlled is too large, there is a priori no reason to expect that a model of the system will lead to a controller stabilizing the true unknown plant. The well-documented possibility of this event is shown in the adaptive closed loop behavior through unacceptable transients ([4], [7]). An example of such a behavior in the case of pole placement of first order systems is postponed to Section 5.5.2 in Chapter 5. In that example, we show that for any arbitrarily chosen integer N, and for any desired stable pole α, and for any initial conditions of the system, there exists an initial guess $\hat{\theta}(0)$ on the real unknown parameter vector θ^0 such that classical adaptive pole placement in α as cited in [72] yields at least N consecutive model-based controllers that are arbitrarily destabilizing the real system. From a practical point of view, a good or bad transient behavior of a control system might be the criterion deciding on the quality of the controller, hence it is crucial to prevent bad transients to occur. This is the reason why the idea of combining classical adaptive control and robust control design appeared ([46],[54], [61], [83], [115]). Rather than designing the controller on the basis of the model irrespective of the model/plant mismatch as it is done in standard certainty equivalence control strategy, one designs a *robust* controller, i.e., a controller that stabilizes any frozen plant in the uncertainty set. However, as it is shown in [46], the requirements of robustness and adaptation often conflict in an adaptive control framework. Therefore, the design of dual controllers, optimal from both estimation and control points of view appears very difficult.

Time-varying model

In an adaptive control design, the model is updated at each iteration. Hence the model-based controller is time-varying, and these time-variations are necessary since they are the key to hopefully further improve the controller performance. Now, let us suppose that the uncertainty on the system to be controlled is initially small - even if this cannot be verified a priori - so that the chosen model at each iteration is controllable and leads to a controller that stabilizes the unknown plant, this at any frozen time of the design. Even in this 'ideal' case, it is well known that the time-varying closed-loop system might not be asymptotically stable if the time variations are too fast [56], [7], [4]. Therefore, loss of asymptotic stability of closed-loop control systems based on the certainty equivalence approach may be induced by the inherent adaptation process they involve. To prevent this phenomenon to occur, the adaptation process should be slow enough to guarantee that the closed-loop system stays within some time-varying stability bounds. Such stability bounds are related to the notion of complex structured stability radius [51]. If the model is chosen so that the Euclidean distance between the certainty equivalence controller and the controller we would obtain on the basis of the unknown system is smaller than the complex structured stability radius of this unknown plant, then time-variations of the model will not affect stability of the closed-loop adaptive system [51], [52]. However, since the system is unknown it is not possible in practice to compute a priori its complex structured stability radius. An interesting approach guaranteeing that the time variations will not destroy the stability of the adaptive scheme is found in [32]: in a switching control system, the switching rate is slowed down so as to avoid switching too fast with respect to the system's settling time, hence destabilizing. This is achieved by adopting a so called *dwell-time switching logic* [48], where a dwell-time is forced between consecutive switching instants. Moreover, this dwell-time is adaptively selected on the basis of available data measurement. According to the results of [49], global stability is secured provided that the switching is *sufficiently slow on average*. In these reported approaches, this interesting concept of dwell-time has been developed for hybrid control systems: the controller parameters are updated when a new estimate of the process parameters becomes available, similarly to the certainty equivalence adaptive control paradigm, but these events occur at discrete instants of time. Moreover, in these approaches, the set of candidate controllers is supposed to be finite and parameterized in a discrete fashion. However, it is not clear how to apply such idea to the case where the controller is continuously parameterized and therefore we will not adopt this approach in the remainder of this thesis.

1.2 The essence of strong robustness

It appears from our discussion in Section 1.1 that classical adaptive control suffers from three drawbacks. An initial insufficient knowledge on the system to control may result in the selection of an uncontrollable model, leading to a paralysis of the control system. Or, the model could lead to a controller that does not stabilize the true plant, in which case undesirable transients may be induced in the closed-loop system behavior. Finally, the time-variations of the model-based controller might destroy the asymptotic stability of the control system. This discussion immediately leads to the following question: how should classical adaptive control schemes be modified so that these three undesirable phenomena are avoided? To the best of our knowledge, each of the three problems discussed in Section 1.1.2 is in itself complicated and so are the corresponding solutions that have been reported in the literature. Hence a direct modification of the estimates obtained by using classical algorithms, combining the various solutions proposed in the literature to these three fundamental problems, so that they meet the three critical properties during adaptation might be a formidable task, and we do not adopt such approach. Instead, our objective is to come up with a different approach, which seeks to overcome the above difficulties, while retaining the advantages and the fundamental ideas on which classical adaptive control is based.

Our idea is as follows. Until the danger of meeting the three problems mentioned above exists, the algorithm would focus on gathering information on the unknown system. Then, when enough information is obtained to guarantee that this danger is avoided, classical adaptive control would be applied. In this line of thought, we ask ourselves the following question. What property should the set of all possible models have so that the three previous undesirable situations cannot occur at any time when performing adaptive control? Clearly, it follows from our previous discussion that the minimum property required during adaptation is the following: the models should keep controllability. Moreover, at each time, the frozen model-based controller should stabilize the real system to be controlled. Finally, adaptation should be slow enough to secure global stability of the adaptive scheme. However, the real system is unknown hence it is a priori not possible to check whether at each time instant the model-based controller stabilizes the true plant. Instead, assuming that at each time an uncertainty level on the system is given, what may be checked is whether at each time instant the model-based controller stabilizes any other model in the uncertainty set. In that respect, our approach is somehow connected to the concept of *robust adaptive control* proposed in [54],

[55], [46]. Because the true system is unknown, how slow adaptation should remain cannot be specified a priori since it depends on the unknown system to be controlled. However, what may be checked is whether the time-varying closed-loop system formed with any system in the uncertainty set and any sequence of controllers in the set of controllers associated to the model class stays stable. When the uncertainty set only contains controllable systems and has this property, we call it *strongly robust* [25], [29]. The combination of identification of a strongly robust uncertainty set and adaptive control design will be called *strongly robust adaptive control* and is our main concern throughout this thesis. As we shall see in Chapter 5 into details, this approach splits in two phases. In the first phase, because no conclusion can be drawn on strong robustness of the uncertainty set, focus is on identification of the model set, in such a way that it will surely become strongly robust. Once strong robustness is achieved, the system switches to the second phase where a classical control design approach based on certainty equivalence is applied. The main philosophy behind our approach may be thus summarized as follows:

"Do not start control before you are sure that the control action will not deteriorate the system performance" .

This idea may be not too far from real life, since we all might have experienced one day a situation where a wrong guess on a system may make us act in a way that contradicts with what we actually want to do.
Now, the notion of strong robustness being introduced and motivated in an adaptive control framework, many questions arise. To begin with, do strongly robust sets of systems exist? A negative answer to this answer would not allow us to go further. If existence of strongly robust sets of systems exist, how to design an identification input yielding a strongly robust uncertainty set in practice? Back to an adaptive control framework, the strongly robust adaptive scheme should decide whether effort has to be put on identification or control on the basis of strong robustness of the uncertainty set. How to construct a criterion to indicate when the strong robustness property is reached? These are the main issues that will guide us throughout this thesis.

1.3 Thesis outline

We now briefly describe the content of each chapter of the thesis. After presenting the mathematical framework of our work in Chapter 2, Chapter 3 deals with the notion of strong robustness as a mathematical object. Further in Chapter 4, an identification problem is examined in a general context. Next, Chapter 5 exploits the results of the previous chapters to develop a general algorithm for strongly robust adaptive control. Finally, in Chapter 7, conclusions and recommendation for further research are given.

Chapter 2 - Mathematical framework

The mathematical ingredients used in this thesis and our working assumptions are presented. In particular the class of systems and the class of the control objectives that we consider are defined. In addition, we discuss two issues on parameter estimation: set-membership identification and orthogonal projection. These two well-known notions will be exploited in Chapter 5.

Chapter 3 - Strong robustness and related notions

The notion of strong robustness is studied as a mathematical property of sets of systems in the class of systems defined in Chapter 2. Various notions such as strong robustness, time-invariant strong robustness, strong quadratic robustness and weak strong robustness are defined and illustrated by simple examples. Further, we investigate the following question: given a set of systems in the class of systems defined in Chapter 1, what conditions this set must satisfy to enjoy the above strong robustness properties? Such conditions can be expressed in a form that involves the complex and real structured stability radii for Schur matrices introduced in [51]. Next, using our previous discussion, we show that around any system in the class of systems defined in Chapter 2, there exists an open strongly robust neighborhood of this set which is a subset of controllable systems. Afterwards, we focus on the following problem: given a set of systems in our class of systems, how can we practically test whether this set is strongly robust or not? Finally, the last section is devoted to the notion of weak strong robustness and we show how this notion may be used in an adaptive control framework.
Chapter 3 is based on [25], [29], [26].

Chapter 4 - Set-membership identification for control

An identification input is designed with the objective to yield a bounded uncertainty set with decreasing size, in the framework of set-membership identification for strongly robust adaptive control. The aim is to identify an uncertainty set which becomes strongly robust in finite time. The key idea is to consider a $2n$-periodic input sequence, and find sufficient conditions on the $2n$ design parameters so that the uncertainty set is bounded. Then, conditions for a decreasing size of the uncertainty set are established. Combining these two sets of conditions, we then explicit the identification input sequence providing a strongly robust uncertainty set. In this chapter, the approach is as follows. The input-output signals are decomposed along two components: the signals we would obtain if they were $2n$-periodic (the steady-state case), and the signals resulting from non-steady state initial conditions and from the modeling error. The input design is then illustrated by means of two simple examples.
Chapter 4 is based on [28].

Chapter 5 - Strongly robust adaptive control

Exploiting the results established in Chapter 3 and Chapter 4, a strongly robust adaptive control system is constructed. This adaptive scheme splits in two phases, the identification phase, where off-line identification is carried on according to the design proposed in Chapter 4, and the control phase, where a certainty equivalence-based strategy is adopted. The switch from the first to the second phase is orchestrated by the strong robustness criterion developed in Chapter 3: as long as no conclusion can be drawn on strong robustness of the uncertainty set, effort is on identification. Once strong robustness is achieved, then control starts. After describing the general scheme of strongly robust adaptive control, analysis of the algorithm

is provided. Then attention is paid on pole placement design. Strongly robust adaptive pole placement is analyzed in details, and illustrated by means of a simulation example.
Chapter 5 is based on [27].

Chapter 6 - Conclusions and further research

Conclusions on our approach are given, with a particular attention to its effectiveness but also limitations. The working assumptions made throughout this work are discussed so as to see if after further investigation they could be relaxed or if they are fundamental to guarantee the presented results. Finally, recommendations of the author for further research are given.

Chapter 2

Mathematical framework

This chapter presents the mathematical ingredients used in the remainder of this thesis. Since our ultimate goal is to develop a control strategy, we first define the class of systems to which our discussion will apply, as well as the class of control objectives we will consider. Further, we introduce a central notion in system identification theory that will be of great relevance in the further chapters of this thesis: set-membership identification.

2.1 Class of objects

The complexity of most of the systems around us defies all attempts to obtain what we would call ”an exact model” of these systems. In addition, while learning about the real system, there are probably some discrepancies between the information to be known and the actual measured information, known as *measurement errors*. Hence, in many applications, one adopts a trade-off between optimality and complexity of the estimated system: one is satisfied with an approximate description of the system, provided that it adequately describes the features of the system one is interested in. This approximation defines *the model*.
Now, to search for an approximate description of a completely unknown system does not make sense, and it is reasonable to assume that the designer has some a-priori information on this system. Such information are usually of two kinds: the model structure and a measure of the discrepancy between this estimated structure and the actual one, *the uncertainty*.
Formally, in a large body of literature devoted to system theory, it is assumed that the dynamical system we are interested in is described in discrete-time by an equation of the form:

$$y(k) = \Psi_{k-1}(\theta^0) + \delta(k), \tag{2.1}$$

where k is the discretized present time, θ^0 denotes the unknown true parameter vector, $y(k)$ represents the available actual data measurement at time k, Ψ_{k-1} is a known operator indicating how the present measurement depends on θ^0 and the previous measurements, and $\delta(k)$ accounts for the uncertainty affected the true system, due to modeling error and measurement error. The system without uncertainty (i.e., $\delta(k) = 0$, $\forall k$) and with an estimated parameter vector (i.e., θ^0 replaced by the model parameter vector θ) is what we call *the model*.

2.1.1 Models and their representations

In this thesis, it is assumed that in the case where the system we are interested in would not be affected by uncertainty (i.e., $\delta(k) = 0$, $\forall k$ in (2.1)), the data measurements would be generated by a discrete-time linear and time invariant SISO system of order n. Hence, the operator Ψ_{k-1} in (2.1) is a linear operator. We define the class of models as follows.

Definition 2.1.1 (Models) *$\mathcal{P}_n$ denotes the set of linear time-invariant systems of order n described in discrete time by the equation:*

$$y(k+1) = \theta^T \phi(k), \ \forall k, \tag{2.2}$$

where $\phi(k)$ represents the regressor vector given by

$$\phi(k) = (-y(k), \cdots, -y(k-n+1), u(k), \cdots, u(k-n+1))^T \in \mathbb{R}^{2n}, \tag{2.3}$$

denoting by u, y the input and output sequences respectively, and where

$$\theta = (a_{n-1}, \cdots, a_0, b_{n-1}, \cdots, b_0)^T \in \mathbb{R}^{2n} \tag{2.4}$$

denotes the parameter vector.

Notation 2.1.2 To keep the notation simple, any model in $\mathcal{P}_n$ described by (2.2) is associated with its parameter vector θ defined in (2.4). In the sequel " $\theta \in \mathcal{P}_n$" should be read as " the system in $\mathcal{P}_n$ parameterized by θ according to (2.2), (2.3), (2.4)".

Now, the description of models in $\mathcal{P}_n$ is not unique and we will use different representations. In particular, any model in $\mathcal{P}_n$ described by (2.2), (2.3), (2.4) has an equivalent description in term of *input/output difference equation* [93] defined as follows.

Definition 2.1.3 (Models in input/output description) *Consider the system defined by* (2.2), (2.3), (2.4). *This system is completely defined by its input/output difference equation:*

$$\mathcal{A}_\theta(\sigma) y = \mathcal{B}_\theta(\sigma) u, \tag{2.5}$$

where σ denotes the shift operator: $\sigma w(k) := w(k+1)$ and the polynomials $\mathcal{A}_\theta \in \mathbb{R}^n[\xi]$ and $\mathcal{B}_\theta \in \mathbb{R}^{n-1}[\xi]$ are given by

$$\begin{aligned} \mathcal{A}_\theta(\xi) &= \xi^n + a_{n-1}\xi^{n-1} + \cdots + a_0 \\ \mathcal{B}_\theta(\xi) &= b_{n-1}\xi^{n-1} + \cdots + b_0. \end{aligned} \tag{2.6}$$

Any model in $\mathcal{P}_n$ described by (2.2), (2.3), (2.4) has an equivalent *input/state/output description* [93] defined as follows.

Definition 2.1.4 (Models in input/state/output description) *Consider the system defined by* (2.2), (2.3), (2.4). *This system is completely defined by its input/state/output description $(A(\theta), B(\theta), C)$ defined as follows:*

$$\begin{aligned} x(k+1) &= A(\theta)x(k) + B(\theta)u(k) \\ y(k) &= Cx(k), \end{aligned} \tag{2.7}$$

where $A(\theta) \in \mathbb{R}^{2n-1\times 2n-1}$, $B(\theta) \in \mathbb{R}^{2n-1}$ *and* $C \in \mathbb{R}^{1\times(2n-1)}$ *are given by*

$$A(\theta) = \begin{bmatrix} -a_{n-1} & \cdots & \cdots & -a_1 & -a_0 & b_{n-2} & \cdots & \cdots & b_1 & b_0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & & \vdots & \vdots & \vdots & & & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots & \vdots & & & \vdots & \vdots \\ \vdots & & & 1 & \vdots & \vdots & & & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & & & 0 & \vdots & 1 & & & \vdots & \vdots \\ \vdots & & & \vdots & \vdots & 0 & \ddots & & \vdots & \vdots \\ \vdots & & & \vdots & \vdots & \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 1 & 0 \end{bmatrix} \tag{2.8}$$

$$B(\theta) = \begin{bmatrix} b_{n-1} & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}^T \tag{2.9}$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \tag{2.10}$$

and the non-minimal state vector $x \in \mathbb{R}^{2n-1}$ *is given by*

$$x(k) = \begin{bmatrix} y(k) & \cdots & y(k-n+1) & u(k-1) & \cdots & u(k-n+1) \end{bmatrix}^T \tag{2.11}$$

In our work, controllable systems in $\mathcal{P}_n$ and asymptotically stable systems in $\mathcal{P}_n$ will play a fundamental role. We now introduce the two induced subsets in $\mathcal{P}_n$.

Definition 2.1.5 (Controllable models and asymptotically stable models) *The set of controllable systems in* $\mathcal{P}_n$ *is denoted by* $\mathcal{C}_n$ *and the set of asymptotically stable systems in* $\mathcal{P}_n$ *is denoted by* $\mathcal{S}_n$*. Controllability here refers to the case where no pole/zero cancellation phenomenon can occur.*

We have the following [93]:

Theorem 2.1.6 (Controllability) *Using the notation introduced in Definition 2.1.3 and Definition 2.1.4 , the following statements are equivalent:*

1. *The system defined by* (2.2), (2.3), (2.4) *is controllable.*
2. $\text{rank}[\lambda I - A(\theta)\ B(\theta)] = 2n-1$, *for all* $\lambda \in \mathbb{C}$.
3. $\text{rank}[B(\theta)\ A(\theta)B(\theta)\ \cdots (A(\theta))^{2n-2}B(\theta)] = 2n-1$.
4. $\gcd(\mathcal{A}_\theta(\xi), \mathcal{B}_\theta(\xi)) = 1$.

Similarly, we have the following stability characterization [93]:

Theorem 2.1.7 (Asymptotic stability) *Using the notation introduced in Definition 2.1.3 and Definition 2.1.4 , the following statements are equivalent:*

1. *The system defined by* (2.2), (2.3), (2.4) *is asymptotically stable.*

2. $A(\theta)$ *is strictly Schur stable, i.e.,* $\det(\lambda I - A(\theta)) = 0, \lambda \in \mathbb{C} \Rightarrow |\lambda| < 1$.

3. *All the roots of* $\mathcal{A}_\theta(\xi)$ *are inside the open unit disc, i.e.,* $\mathcal{A}_\theta(\lambda) = 0, \lambda \in \mathbb{C} \Rightarrow |\lambda| < 1$.

Notation 2.1.8 In accordance with Notation 2.1.2, we will use the notation $\theta \in \mathcal{C}_n$ (respectively $\theta \in \mathcal{S}_n$) to refer to the system defined by (2.2), (2.3), (2.4) under the assumption that it is controllable (respectively asymptotically stable).

2.1.2 Actual system

We now describe the complete true system to be controlled, relaxing the assumption that the uncertainty δ in (2.1) is zero.

Definition 2.1.9 (Actual system) *We assume that the system we are interested in, from which the input-output measurements are obtained, is described by:*

$$y(k+1) = (\theta^0)^T \phi(k) + \delta(k), \ \forall k, \tag{2.12}$$

where for all k, $y(k)$ *is the actual measured output at time* k, $\theta^0 \in \mathcal{C}_n \cap \mathcal{S}_n$ *is the unknown system model of the form* (2.4), ϕ *is the regressor vector given in* (2.3) *composed of known actual measurement input-output data* $u(i), y(i)$,$i \leq k$ *and* $\delta(k)$ *is the uncertainty at time* k.

In this thesis, we will make the following assumption on the uncertainty sequence δ.

Assumption 2.1.10 (Unknown-but-bounded uncertainty) *The uncertainty sequence* δ *in* (2.12) *is unknown-but-bounded with a known bound, i.e., there exist two real constants* $\overline{\delta}, \underline{\delta}$ *such that* $\underline{\delta} \leq \overline{\delta}$ *and* $\underline{\delta} \leq \delta(k) \leq \overline{\delta}$, $\forall k$.

Remark 2.1.11 Assumption 2.1.10 is often simplified in the literature by taking $|\delta(k)| \leq \delta_1$, $\forall k$, where $\delta_1 = \max\{|\underline{\delta}|, |\overline{\delta}|\}$, which only increases the conservatism of the upper and lower bounds on δ.

Remark 2.1.12 In the description (2.12), the true parameter vector θ^0 corresponds to a system in $\mathcal{P}_n$ which is controllable and asymptotically stable. The motivation of these two assumptions, as well as the motivation of Assumption 2.1.10 on the structure of the uncertainty, are postponed to Chapter 5 (see Remark 5.2.4).

2.1.3 Controllers

In this thesis the main goal is to discuss a control problem, and therefore the class of controllers we are going to consider is one of the central notions in our work. The notion of *control objective* is taken in its wide sense, that is to improve performance of the considered system (2.12), in a way that is left unspecified for the moment. However, we make the following assumption.

Assumption 2.1.13 (Controllers) *There exists a single-valued continuous map*

$$f : \theta \in \mathcal{C}_n \longmapsto f(\theta) \in \mathbb{R}^{1\times(2n-1)} \tag{2.13}$$

that assigns any model in $\mathcal{C}_n$ *defined by* (2.2)*,* (2.3) *and* (2.4) *with its controller* $f(\theta) \in \mathbb{R}^{1\times(2n-1)}$ *leading to the control law*

$$u(k) = f(\theta)x(k), \forall k, \tag{2.14}$$

where x *is the state vector defined in* (2.7)*, such that the closed-loop system defined by*

$$\begin{aligned} x(k+1) &= (A(\theta) + B(\theta)f(\theta))x(k) \\ y(k) &= Cx(k) \end{aligned} \tag{2.15}$$

is asymptotically stable, i.e., the dynamic matrix $A(\theta) + B(\theta)f(\theta)$ *is strictly Schur stable.*

Remark 2.1.14 The assumption that the map f is continuous is motivated later in this thesis (see Chapter 3, Theorem 3.2.12).

More specifically, pole placement in stable poles and linear quadratic control will be the control objectives that will be mainly considered. For this reason, we now give a brief overview of these two control design approaches and show that both satisfy Assumption 2.1.13.

Pole placement in stable poles

Consider $\theta \in \mathcal{C}_n$ as the system to be control. The problem of pole placement in some stable poles consists of designing an input law of the type (2.14) such that the poles of the resulting closed-loop system (2.15) (the eigenvalues of the matrix $A(\theta)+B(\theta)f(\theta)$), are located in the roots of a pre-specified strictly Schur-stable *desired closed-loop characteristic polynomial* of the form:

$$\Pi(\xi) = \prod_{i=1}^{2n-1} (\xi - \alpha_i), \tag{2.16}$$

with

$$|\alpha_i| < 1, \ \forall i = 1, \cdots, 2n-1. \tag{2.17}$$

For a given system defined by (2.2), (2.3) and (2.4) such that $\theta \in \mathcal{C}_n$, this control objective is achieved by a unique controller $f(\theta)$ given by [93]:

$$u(k) = F(A(\theta), B(\theta))x(k), \forall k, \tag{2.18}$$

where $x(k)$ is given in (2.7), $A(\theta)$ and $B(\theta)$ are given in (2.9), (2.10) and

$$F : \{(A, B) \in \mathbb{R}^{(2n-1)\times(2n-1)} \times \mathbb{R}^{(2n-1)\times 1} : (A, B) \text{ is controllable } \} \rightarrow \mathbb{R}^{1\times 2n-1} \tag{2.19}$$

is defined by Ackermann's Formula [72]:

$$F(A, B) = -[0 \ \cdots \ 0 \ 1][B \ AB \ \cdots \ A^{2n-2}B]\Pi(A), \tag{2.20}$$

where Π is the desired closed-loop polynomial defined in (2.16). The closed loop system is hence defined by:

$$\begin{aligned} x(k+1) &= (A(\theta) + B(\theta)F(A(\theta), B(\theta)))x(k) \\ y(k) &= Cx(k), \end{aligned} \tag{2.21}$$

having its poles exactly in the desired poles α_i, $i = 1, \cdots, 2n-1$. Hence it follows from (2.17) that the closed-loop system (2.21) is asymptotically stable. Finally, it follows from the expression (2.20) that the map $f : \mathcal{C}_n \to \mathbb{R}^{1\times(2n-1)}$ assigning to the system parameter θ the control gain $f(\theta) = F(A(\theta), B(\theta))$ is continuous. Therefore, the problem of pole placement in stable poles defined by (2.16) and (2.17) satisfies Assumption 2.1.13.

Linear quadratic control

Consider $\theta \in \mathcal{C}_n$ as the system to be controlled. The Linear Quadratic Control problem applied to the system (2.7) consists of designing an input law of the type (2.14) which minimizes the performance index:

$$J(u, x(0)) = \sum_{k=0}^{\infty} (x(k))^T C^T C x(k) + \rho(u(k))^2, \tag{2.22}$$

where $x(k)$ is given in (2.11), $C \in \mathbb{R}^{1\times(2n-1)}$ is given in (2.10) and $\rho > 0$ is fixed by the designer. The solution of this problem is unique and given by [7]:

$$u(k) = f(\theta)x(k), \forall k, \tag{2.23}$$

where $f(\theta)$ is given by

$$f(\theta) = -\frac{1}{\rho}(B(\theta))^T P(\theta), \tag{2.24}$$

where $P(\theta) = (P(\theta))^T \geq 0$ is the unique positive semi-definite solution in $\mathbb{R}^{(2n-1)\times(2n-1)}$ solution of the algebraic Riccati equation:

$$(A(\theta))^T P(\theta) + P(\theta)A(\theta) + C^T C - \frac{1}{\rho}P(\theta)B(\theta)(B(\theta))^T P(\theta) = 0, \tag{2.25}$$

with $A(\theta)$ and $B(\theta)$ as defined in (2.9) and (2.10) respectively. Moreover, the closed system (2.15) is asymptotically stable, i.e., the eigenvalues of the matrix $A(\theta) - \frac{1}{\rho}B(\theta)(B(\theta))^T P(\theta)$ are in the interior of the unit disc. It follows from (2.24), (2.25) that the map $f : \mathcal{C}_n \to \mathbb{R}^{1\times(2n-1)}$ assigning to the system parameter θ the control gain $f(\theta) = -\frac{1}{\rho}(B(\theta))^T P(\theta)$ is continuous. Hence the Linear quadratic control problem with the performance criterion given in (2.22) satisfies Assumption 2.1.13.

2.2 Set-membership identification

As discussed in Chapter 1, adaptive control typically deals with partially unknown systems. When the uncertainty on this system to be controlled is small enough, certainty equivalence control design is generally adopted, i.e., the identified model is used for on-line controller without any regard for the model errors [7], [72], [54]. However, if the uncertainty level is unknown, a preferred approach consists in gathering information on the system through input-output measurements, so as to reduce the uncertainty level recursively, until certainty equivalence can be applied. In this case, rather than estimating a single model, one identifies the set of all model candidates, i.e., the models that are consistent with all the available data measurements. This method, called *set-membership identification* and introduced in [104], has become a central issue in identification theory.

2.2.1 Membership set: computation

Contrary to identification methods involving point estimation of a single model, set membership identification consists of estimating the set containing all the models with a given structure that are consistent with the available data measurements, the model structure and the prior knowledge on the uncertainty, *the membership set*.
Clearly, the way this set is computed depends on the assumed model structure ([16], [17], [80], [84]) and on the characteristics of the uncertainty ([9], [109], [110]). However, a large number of approaches in set-membership identification literature consider time-invariant SISO linear systems and assume the uncertainty to be unknown-but-bounded with known bounds, according to Assumption 2.1.10 ([11], [12], [9], [39], [78], [105], [81]).

Let us suppose the system under consideration to be of the form (2.12). Suppose that the uncertainty sequence δ satisfies Assumption 2.1.10. We have that all system models in $\mathcal{P}_n$ consistent with the kth measurement $(y(k), \phi(k-1))$ belong to the set

$$\mathcal{G}(k) = \{\theta \in \mathcal{P}_n : \underline{\delta} \leq y(k) - \theta^T \phi(k-1) \leq \overline{\delta}\}, \tag{2.26}$$

This set $\mathcal{G}(k)$ is the hyperstrip in $\mathbb{R}^{2n}$ bounded by the two parallel hyperplanes:

$$\underline{\mathcal{H}}(k) = \{\theta \in \mathcal{P}_n : y(k) - \theta^T \phi(k-1) = \underline{\delta}\} \tag{2.27}$$

$$\overline{\mathcal{H}}(k) = \{\theta \in \mathcal{P}_n : y(k) - \theta^T \phi(k-1) = \overline{\delta}\} \tag{2.28}$$

Hence, for a finite number of given measurements $(y(i), \phi(i-1))_{i=1,\cdots,k}$, the membership set is given by

$$\hat{\mathcal{G}}(k) = \bigcap_{i=1}^{k} \mathcal{G}(i), \tag{2.29}$$

where $\mathcal{G}(i)$, $i = 1, \cdots, k$ is given in (2.26). It is worth to note that a Matlab toolbox, *the Geometric Bounding Toolbox* [107] is available for the computation of the membership set given in (2.29).

2.2.2 Membership set: properties

We now focus on various properties of the membership set given by (2.29).

Convexity and closeness

An interesting property in set-membership identification is that the membership-set computed according to (2.29) is convex and closed. Now, if the system to be identified satisfies Assumption 2.1.9, then it is controllable and asymptotically stable, i.e., the unknown system parameter θ^0 is element of $\mathcal{C}_n \cap \mathcal{S}_n$. Therefore it seems that a "good" estimate should also have these two properties, in which case they should be within the set $\hat{\mathcal{G}}^*$ of parameters that are consistent with all the available measurements and the prior knowledge on the real system defined by

$$\hat{\mathcal{G}}^*(k) = \hat{\mathcal{G}}(k) \cap \mathcal{C}_n \cap \mathcal{S}_n, \forall k. \tag{2.30}$$

However, because the sets $\mathcal{C}_n$ and $\mathcal{S}_n$ of controllable and asymptotically stable systems in $\mathcal{P}_n$ are nor convex [114] neither closed in general, the set $\hat{\mathcal{G}}^*(k)$ defined in (2.30) might not be convex nor closed. Yet most parameter estimation methods are based on convex minimization problems, such as the least squares identification algorithm [63], the gradient projection algorithm [54], or the orthogonal projection algorithm [15]. Hence convexity and closeness of the model set are desirable properties, which is the reason why one is often satisfied with an estimate $\theta \in \hat{\mathcal{G}}(k)$, $\forall k$, even if this model is not controllable or asymptotically stable. Then, of course, if the control design involves this model, the model has to be replaced by a controllable one. This matter is further discussed in Chapter 3 (Section 3.3.1).

Width of hyperstrips

The width of the hyperstrip $\mathcal{G}(k)$ containing all the parameter vectors consistent with a given measurement $(y(k), \phi(k-1))$ is a function of the uncertainty on this measurement.
Suppose the system to be of the type (2.12), such that the uncertainty sequence δ satisfies Assumption 2.1.10. Provided that $||\phi(k-1)|| \neq 0$, the width of the hyperstrip $\mathcal{G}(k)$ obtained on the basis of the measurement $(y(k), \phi(k-1))$ and defined in (2.26) is given by:

$$\mathcal{W}(k) = \frac{\overline{\delta} - \underline{\delta}}{||\phi(k-1)||}. \tag{2.31}$$

Remark 2.2.1 If the uncertainty is zero, i.e., $\delta = 0$, then the hyperstrip $\mathcal{G}(k)$ computed in (2.26) is reduced to a hyperplane, i.e., (2.31) is replaced by $\mathcal{W}(k) = 0$.

The following result immediately follows from (2.31).

Property 2.2.2 *Consider any given system of the form* (2.12) *such that the uncertainty satisfies Assumption 2.1.10.Then we have that*

$$\textit{if } \lim_{k\to\infty} ||\phi(k)|| = \infty \textit{ then } \lim_{k\to\infty} \mathcal{W}(k) = 0, \tag{2.32}$$

where $\mathcal{W}(k)$ is given in (2.31).

Boundedness of the membership set

Since the membership-set is computed as the intersection of hyperstrips in $\mathbb{R}^{2n}$, it must be obtained on the basis of at least $2n$ non-parallel hyperstrips. Now, noticing that two hyperstrips $\mathcal{G}(i)$ and $\mathcal{G}(j)$, $i \neq j$, are not parallel if and only if $\phi(i)$ and $\phi(j)$ are linearly independent, we have the following property.

Property 2.2.3 *The set $\hat{\mathcal{G}}(k)$ defined in* (2.29) *is bounded if and only if the two following statements hold:*

i. $k \geq 2n$*;*

ii. there exist $2n$ disctinct integers $k_i \leq k$, $i = 1, \cdots, 2n$, and such that

$$\det([\phi(k_1) \ \cdots \ \phi(k_{2n})]) \neq 0. \tag{2.33}$$

Equation (2.33) is an excitation-type condition.

Radius

We suppose that the uncertainty is structured according to Assumption 2.1.10. After k measurements $(y(i), \phi(i-1))$, $i = 1, \cdots, k$, for any $k \geq 1$, an outer bounding ball for $\hat{\mathcal{G}}(k)$ can be estimated as follows. For all $\theta \in \hat{\mathcal{G}}(k)$, and for all $i \leq k$, we have:

$$\underline{\delta} \leq y(i) - \theta^T \phi(i-1) \leq \overline{\delta}. \tag{2.34}$$

Since $\theta^0 \in \mathcal{G}(k)$, $\forall k$, then (2.34) is satisfied by θ^0. Hence, for all $\theta \in \hat{\mathcal{G}}(k)$, and for all $i \leq k$, we have:

$$\underline{\delta} - \overline{\delta} \leq (\theta^0 - \theta)^T \phi(i-1) \leq \overline{\delta} - \underline{\delta}. \tag{2.35}$$

Denoting by $\tilde{\theta} = \theta^0 - \theta$ the model error, and introducing the notation

$$\underline{\delta}_1 = \overline{\delta} - \underline{\delta} \geq 0, \tag{2.36}$$

(2.35) can be rewritten as:

$$|\tilde{\theta}^T \phi(i-1)| \leq \overline{\delta}_1. \tag{2.37}$$

Therefore, for all $\theta \in \hat{\mathcal{G}}(k)$, and for all $i \leq k$, if $||\phi(i-1)||.\cos(\tilde{\theta}, \phi(i-1)) \neq 0$ then

$$||\tilde{\theta}|| \leq \frac{\overline{\delta_1}}{||\phi(i-1)||.|\cos(\tilde{\theta}, \phi(i-1))|} = \frac{\mathcal{W}(i)}{|\cos(\tilde{\theta}, \phi(i-1))|}, \tag{2.38}$$

where $\mathcal{W}(i)$ is given by (2.31). Hence, if

$$\max_{i \leq k}\{||\phi(i-1)||.\cos(\tilde{\theta}, \phi(i-1))\} \neq 0, \tag{2.39}$$

then the parameter vector θ lies in the ball with center θ^0 and radius $\rho(k)$ where

$$\rho(k) = \min_{i \leq k} \frac{\mathcal{W}(i)}{|\cos(\tilde{\theta}, \phi(i-1))|}. \tag{2.40}$$

Geometrical interpretation : (2.38) can be interpreted geometrically (see Figure 2.1) as follows. For any $i \geq 1$, let θ_i be an element of $\mathcal{G}(i)$ and $\tilde{\theta}_i = \theta_i - \theta^0$. Let $\mathcal{L}(\theta_i)$ denote the hyperline going through θ_i and θ^0. Note that the vector $\phi(i-1)$ is normal to the hyperplanes $\overline{\mathcal{H}}(i)$ and $\underline{\mathcal{H}}(i)$ defined in (2.27). The condition $||\phi(i-1)||.\cos(\tilde{\theta}_i, \phi(i-1)) \neq 0$ is satisfied if an only if $||\phi(i-1)|| \neq 0$ and the vector $\tilde{\theta}_i$ is not normal to $\phi(i-1)$. Provided that this condition holds, the quantity $\rho_i = \mathcal{W}(i)(|\cos(\tilde{\theta}_i, \phi(i-1))|)^{-1}$ in (2.38) represents the largest distance between the two points θ_i and θ^0, expressing the constraint that θ_i and θ^0 are in $\mathcal{L}(\theta_i) \cap \mathcal{G}(i)$. Finally, in (2.40), the quantity $\rho(k) = \min_{i \leq k} \mathcal{W}(i)(|\cos(\tilde{\theta}, \phi(i-1))|)^{-1}$ represents the largest distance between two points θ_i and θ^0, expressing the constraint that θ_i and θ^0 are in $\mathcal{L}(\theta_i) \cap \mathcal{G}(i)$, $\forall i \leq k$.

Note that the constant $\overline{\delta}_1$ defined in (2.36) is known and the term $\phi(i-1)$ is measured at any time $i \leq k$. However, since $\tilde{\theta}$ is unknown, the expression of $\rho(k)$ in (2.40) is not computable a-priori. However, the result (2.40) clearly shows that if the condition (2.39) holds for all $k \geq 1$, then we have

$$\text{if } \lim_{k \to \infty} \mathcal{W}(k) = 0, \text{ then } \lim_{k \to \infty} \rho(k) = 0. \tag{2.41}$$

Hence, the following property follows from Property 2.2.2.

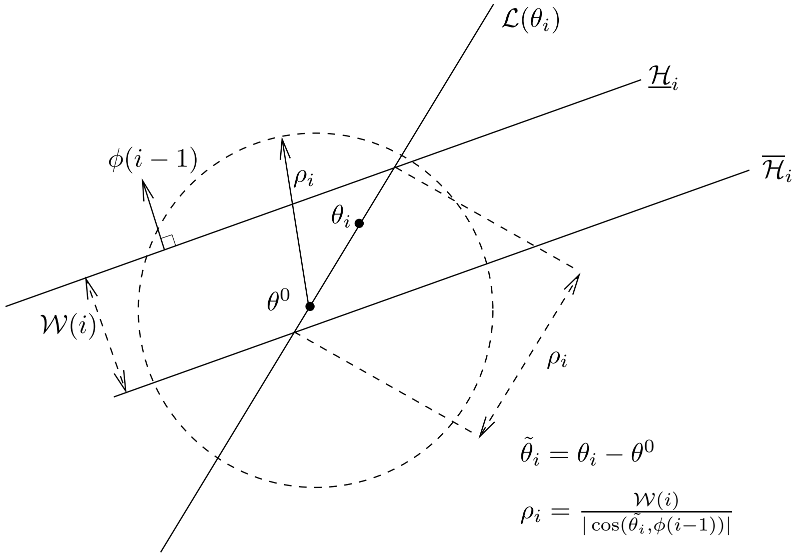


Figure 2.1: Membership set: outer bounding sphere

Property 2.2.4 *Consider any given system of the form* (2.12) *such that the uncertainty satisfies Assumption 2.1.10. Suppose the input sequence to be such that that the condition* (2.39) *holds for all* $k \geq K$*,* $K > 0$*. Then we have:*

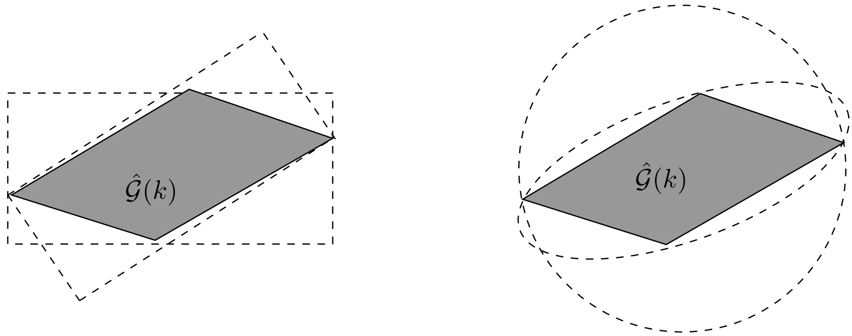
$$\textit{if } \lim_{k\to\infty} ||\phi(k)|| = \infty \textit{ then } \lim_{k\to\infty} \rho(k) = 0, \tag{2.42}$$

where $\mathcal{W}(k)$ *is given in* (2.31).

Property 2.2.4 indicates that if the identification input sequence satisfies the condition (2.39) and yields regressors with arbitrarily large magnitude, then the membership set $\hat{\mathcal{G}}(k)$ converges to the point set $\{\theta^0\}$.

Outer bounding sets

A Matlab toolbox is now available to compute the exact polytopic membership set $\hat{\mathcal{G}}(k)$ given in (2.29) on the basis of measurements [107]. However, this recursive computation is quite cumbersome and rather often, a tight approximation of this set is sufficient to describe the set of models that are consistent with the available data measurements [59], [5]. Hence, many approaches are based on the computation of an outer-bounding set of $\hat{\mathcal{G}}(k)$, leading to easier computation. Of course, this outer bounding should be ”tight” in order to be a good approximation of $\hat{\mathcal{G}}(k)$, with a minimal size or volume. This idea, illustrated in Figure 2.2, gave rise to the concept of *optimal* bounding sets, such as optimal bounding orthotopes [79], or the more popular *optimal bounding ellipsoids* (OBE) [39], [34], [98]. The recursive computation of optimal bounding ellipsoids can be done according to various *ellipsoid algorithms* [100], [23], [19], [66], [58], [98]. The central idea of these algorithms is the computation of a



Outer bounding orthotopes Outer bounding ellipsoids

Figure 2.2: Membership set: outer bounding sets

supporting halfspace of one constraint set. When intersecting with the current ellipsoidal approximation to the membership set, the halfspace should provide maximal volume reduction in the approximation. It can be shown that good convergence properties of these algorithms would imply that the outer-bounding ellipsoids asymptotically approach the membership set in some meaningful sense. Moreover, it has been shown that the centroid of these ellipsoids can be derived as a solution of a certain constrained least square problem that is computationally cheap.

To illustrate this concept of outer bounding ellipsoidal approximation of the membership set in our framework, let us suppose that we have obtained $2n$ distinct measurements $(y(k_i), \phi(k_i - 1))$, $i = 1, \cdots, 2n$, and $k_1 < k_2 < \cdots < k_{2n}$. Let us suppose the boundedness condition (2.33) is satisfied, implying that the membership set $\hat{\mathcal{G}}(k_{2n})$ is bounded.

Initial outer bounding ellipsoid: the first step of the classical ellipsoid algorithm would consist in computing an ellipsoidal set which contains $\hat{\mathcal{G}}(k_{2n})$, corresponding to the following problem.

Problem 2.2.5 *Find* (P, ω) *with* $P = P^T \in \mathbb{R}^{2n \times 2n}$ *and* $\omega \in \mathbb{R}^{2n}$ *such that*

$$\hat{\mathcal{G}}(k_{2n}) \subset \{\theta : (\omega - \theta)^T P^{-1} (\omega - \theta) \leq 1\}. \tag{2.43}$$

Such a solution always exists, since $\hat{\mathcal{G}}(k_{2n})$ is bounded, and the set

$$\mathcal{E} = \{\theta : (\omega - \theta)^T P^{-1} (\omega - \theta) \leq 1\} \tag{2.44}$$

is then an outer bounding ellipsoid for $\hat{\mathcal{G}}(k_{2n})$. ω is the center of the ellipsoid $\mathcal{E}$ and the positive definite matrix P gives the ”size” and orientation of $\mathcal{E}$ [23]: the square roots of the eigenvalues of P are the lengths of the semi-axes of $\mathcal{E}$. The volume of $\mathcal{E}$ is given by

$$\text{Vol}(\mathcal{E}) = \mathcal{V}_0 \sqrt{\det(P)}, \tag{2.45}$$

where $\mathcal{V}_0$ denotes the volume of the unit ball in $\mathbb{R}^{2n}$.

Recursive optimal outer bounding ellipsoid: at each new measurement $(y(k), \phi(k-1))$, $k \geq k_{2n} + 1$, a new ellipsoid bounding the intersection $\hat{\mathcal{G}}(k-1) \cap \mathcal{G}(k)$ is computed, where $\mathcal{G}(k)$ is given in (2.26). Since $\mathcal{G}(k)$ can be seen as the intersection of the two halfplanes

$$H_1 = \{\theta \in \mathbb{R}^{2n} : y(k) - \theta^T \phi(k-1) \geq \underline{\delta}\} \tag{2.46}$$

$$H_2 = \{\theta \in \mathcal{P}_n : y(k) - \theta^T \phi(k-1) \leq \overline{\delta}\} \tag{2.47}$$

the problem is to compute an outer bounding ellipsoid for the intersection of the ellipsoid $\mathcal{E}$ defined in (2.44) and the two closed halfplanes H_1 and H_2. In this respect we have the following result [41].

Lemma 2.2.6 (Minimum volume bounding ellipsoid) *Let $N \geq 2$ and let $\mathcal{E} \subset \mathbb{R}^N$, be an ellipsoid with center ω and described by the positive definite matrix P. Also, let H be the closed halfspace $\{x : a^T x \leq \beta\}$, $a \in \mathbb{R}^N$ and $\beta > 0$. The minimum volume ellipsoid $\hat{\mathcal{E}}$ bounding the intersection $\mathcal{E} \cap H$ is described by*

$$\hat{\mathcal{E}} = \{x \in \mathbb{R}^{2n} : (x - \hat{\omega})^T \hat{P}^{-1} (x - \hat{\omega}) \leq 1\} \tag{2.48}$$

$$\hat{\omega} = \omega - \tau \frac{Pa}{\sqrt{a^T Pa}}, \; \hat{P} = \mu(P - \kappa \frac{Paa^T P}{a^T Pa}) \tag{2.49}$$

where

$$\tau = \frac{1 + N\alpha}{N + 1}, \; \alpha = \frac{a^T \omega - \beta}{\sqrt{a^T Pa}}, \; \mu = \frac{N^2 (1 - \alpha^2)}{N^2 - 1}, \; \kappa = \frac{2(1 + N\alpha)}{(N + 1)(1 + \alpha)}. \tag{2.50}$$

For $\alpha > 1$, $\mathcal{E} \cap H = \emptyset$ and for $\alpha \leq -1/N$, $\hat{\mathcal{E}} = \mathcal{E}$. For $-1/N < \alpha < 1$, the ratio of the volume of $\hat{\mathcal{E}}$ to the volume of $\mathcal{E}$ is a decreasing function and is given by

$$r(\alpha) = (\mu^N (1 - \kappa))^{1/2}, \tag{2.51}$$

where μ and κ are given in (2.50).

Lemma 2.2.6 provides a way to compute recursively the smallest ellipsoid outer-bounding the membership set.

2.2.3 Model selection

The purpose of set-membership identification is to provide us with the set of all model candidates to represent the real system. Now, in a more general adaptive control scheme, one usually desires to update a model point, on the basis of which the controller is designed using the certainty equivalence principle. Supposing that the membership set is computed at each measurement, it is natural to choose this model in the membership set, in such a way that if the new measurement does not falsify the present model, then this model is not updated. Conversely, if the new measurement does falsify the present model, then the model is updated in a new model which is closer to the real unknown system. A large part of the approaches following this idea include the celebrated *least squares* (LS) or, more generally, the *weighted least squares* (WLS) algorithm [2], [77].

Least squares algorithm

For a given set of data $(y(i), \phi(i-1))$, $i = 1, \cdots, k$, the WLS estimate $\hat{\theta}_k$ is the solution of the minimization problem

$$\hat{\theta}_k = \arg\min_{\theta} \sum_{i=1}^{k} q_i (y(i) - \theta^T \phi(i-1))^2 \quad (2.52)$$

where the terms q_i are nonnegative weights. This algorithm has the great advantage to be implemented recursively as follows. Given the estimate at time k, $\hat{\theta}_k$, and the new measurement $(y(k+1), \phi(k)$, the new RWS estimate is computed by

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \frac{q_{k+1} P_k \phi(k)}{1 + q_{k+1} (\phi(k))^T P_k \phi(k)} (y(k+1) - \hat{\theta}_k^T \phi(k)), \forall k, \quad (2.53)$$

where the matrix $P_k \in \mathbb{R}^{2n \times 2n}$ is computed recursively by

$$P_{k+1} = P_k - \frac{q_{k+1} P_k \phi(k) (\phi(k))^T P_k}{1 + q_{k+1} (\phi(k))^T P_k \phi(k)}, \forall k. \quad (2.54)$$

Another advantage of this WLS estimate recursive computation is that it does not need any a-priori assumption on the uncertainty δ in (2.12). However, it is well known that the WLS estimate given in (2.52) is in general not in the membership set $\hat{\mathcal{G}}(k)$. Therefore, some modification is necessary to ensure that the WLS estimate lies in or converges to the membership set. In this respect, the authors of [10] established the following result in the case of bounded-but-unknown uncertainty.

Theorem 2.2.7 (Modified Recursive Least squares) *consider the system* (2.12) *and suppose that the uncertainty sequence δ satisfies Assumption 2.1.10 with $-\underline{\delta} = \overline{\delta} \geq 0$. Consider the recursive WLS algorithm* (2.53) *and* (2.54) *with $P_0 = P_0^T$ and arbitrary $\hat{\theta}_0$. For any $\epsilon > 0$, let q_k be defined for all $k \geq 1$ by:*

$$\begin{aligned} q_k &= 0, \; |y(k) - (\hat{\theta}_{k-1})^T \phi(k-1)| \leq \overline{\delta} + \epsilon, \\ q_k &= \frac{|y(k) - (\hat{\theta}_{k-1})^T \phi(k-1)| - \overline{\delta}}{\overline{\delta} (\phi(k-1))^T P_{k-1} \phi(k)}, \; |y(k) - (\hat{\theta}_{k-1})^T \phi(k-1)| > \overline{\delta} + \epsilon. \end{aligned} \quad (2.55)$$

Then the WLS estimate $\hat{\theta}_k$ converges to the membership set asymptotically in the following sense: for any $\epsilon > 0$, there exists a finite number $N_\epsilon \in \mathbb{N}$ such that $\forall i \geq N_\epsilon$,

$$\hat{\theta}_k \in \bigcap_{m=N_\epsilon}^{\infty} \{\theta \in \mathbb{R}^{2n} : |y(m) - \theta^T \phi(m-1)| \leq \overline{\delta} + \epsilon\}. \quad (2.56)$$

Yet in some applications, asymptotic convergence of the estimate to the membership set might not be satisfactory. In particular, in this thesis (Chapter 5) it will be crucial that the estimate $\hat{\theta}_k$ belongs to the membership set. Hence, we will use the orthogonal projection algorithm.

Orthogonal projection algorithm

For a given set of data $(y(i), \phi(i-1))$, $i = 1, \cdots, k$, $k \geq 1$, the orthogonal projection estimate $\hat{\theta}_k$ is computed as the orthogonal projection of the previous estimate on the present membership set:

$$\hat{\theta}_k = \arg \min_{\theta \in \hat{\mathcal{G}}(k-1)} (\theta - \hat{\theta}_{k-1})^T (\theta - \hat{\theta}_{k-1}). \tag{2.57}$$

Figure 2.3 illustrates this idea. This algorithm guarantees that the estimate is in the membership set, at any time. If the estimate is updated according to (2.57), then we have the following property [72].

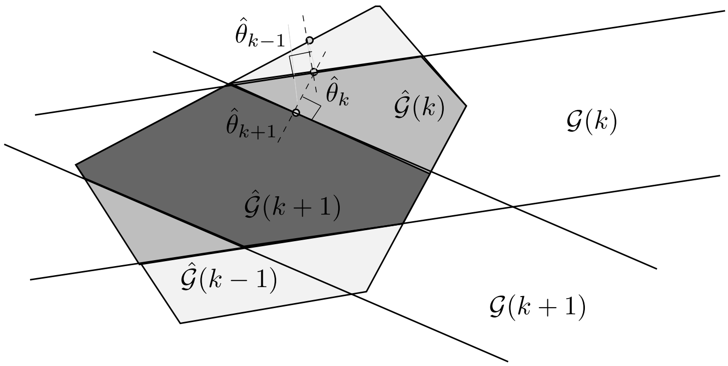


Figure 2.3: Orthogonal projection algorithm

Property 2.2.8 (Orthogonal projection algorithm) *The model error sequence is non-increasing, i.e.,*

$$||\theta^0 - \hat{\theta}_{k+1}|| \leq ||\theta^0 - \hat{\theta}_k||, \ \forall k, \tag{2.58}$$

and is asymptotically slow, i.e.,

$$\lim_{k \to \infty} ||\hat{\theta}_{k+1} - \hat{\theta}_k|| = 0. \tag{2.59}$$

Property 2.2.8 holds regardless how the input is generated.

Chapter 3

Strong robustness and related notions

In this chapter the notion of strong robustness introduced in Chapter 1 is mathematically defined. Then, issues raised by the introduced concept are investigated within the mathematical framework defined in the previous chapter and some of the results are illustrated by means of simple examples in the first order case. In particular it is established that there exists a strongly robust open neighborhood around any systems in $\mathcal{C}_n$. Moreover, it is proved that if a set of systems satisfies a criterion involving the notion of structured stability radius, then this set is strongly robust with respect to any control objective belonging to the previously defined class of control objectives. Further, for specified subsets of systems in $\mathcal{C}_n$, we show that tractable tests for characterizations of strong robustness can be constructed using linear matrix inequalities (LMI's) or a Kharitonov-like stability test. Finally, an extended notion of strong robustness, called weak strong robustness, is investigated.

3.1 Definitions

3.1.1 Strong robustness

We first recall the definitions of asymptotic stability and quadratic stability for linear discrete systems.

Definition 3.1.1 (Asymptotic and quadratic stability) *Consider the linear time-varying discrete system described by*

$$x(k+1) = M(k)x(k),\ x(0) \tag{3.1}$$

where $x \in \mathbb{R}^N$ denotes the state vector and $M(k) \in \mathbb{R}^{N\times N}$ denotes the dynamics matrix.
1. The system (3.1) *is asymptotically stable [93] if there exists $\epsilon > 0$ such that if $||x(0)|| \leq \epsilon$, then* $\lim_{k\to\infty} ||x(k)|| = 0$.
2. The system (3.1) *is quadratically stable [111] if there exists a matrix $K = K^T \in \mathbb{R}^{N\times N}$ such that $K > 0$ and*

$$[M(k)]^T KM(k) - K + I < 0,\ \forall k. \tag{3.2}$$

We define the notion of *strong robustness* as follows.

Definition 3.1.2 (Strong robustness) *Let Ω be a subset of the set $\mathcal{P}_n$ defined in Definition 2.1.1. Suppose that the control objective is given and satisfies Assumption 2.1.13. For any system $\theta \in \mathcal{C}_n$ defined by*

$$y(k+1) = \theta^T \phi(k), \phi(0),$$

where

$$\theta = (a_{n-1} \ \cdots \ a_0 \ b_{n-1} \ \cdots \ b_0)^T \in \mathcal{C}_n, \tag{3.3}$$

and where the regressor vector ϕ is given by

$$\phi(k) = (-y(k) \ \cdots \ -y(k-n+1) \ u(k) \ \cdots \ u(k-n+1))^T \in \mathbb{R}^{2n}, \tag{3.4}$$

$f(\theta)$ denotes the controller defined in Assumption 2.1.13, leading to the control law

$$u(k) = f(\theta)x(k), \tag{3.5}$$

where the state vector x is given by

$$x(k) = (y(k), \cdots, y(k-n+1), u(k-1), \cdots, u(k-n+1))^T. \tag{3.6}$$

The set of systems Ω is strongly robust with respect to the given control objective if the two following conditions hold:

- *$\Omega \subset \mathcal{C}_n$;*
- *for any system $\theta \in \Omega$ and for any sequence of systems $\{\theta(k)\}_{k \in \mathbb{N}} \subset \Omega$, the time-varying closed-loop system defined by:*

$$\begin{aligned} y(k+1) &= \theta^T \phi(k) \\ u(k) &= f(\theta(k))x(k), \end{aligned} \tag{3.7}$$

  where the vectors $\phi(k) \in \mathbb{R}^{2n}$ and $x(k) \in \mathbb{R}^{2n}$ are given in (3.4) *and* (3.6) *respectively, is asymptotically stable as defined in Definition 3.1.1.*

Remark 3.1.3 Definition 3.1.2 could be extended to a larger class of systems. Indeed, a similar definition could be established considering systems that may be nonlinear, time-varying, in continuous time description, stochastic and non-asymptotically stable. However, in this thesis we restrict ourselves to sets of systems in $\mathcal{P}_n$.

Remark 3.1.4 The main difference between the classical concept of robustness and the strong robustness notion lies in the idea that the former involves a fixed nominal model, whereas the latter allows time-variability.

Property 3.1.5 The following statements hold:

- Any subset of a strongly robust set of systems is strongly robust.
- For any system $\theta \in \mathcal{C}_n$, the point-set $\{\theta\}$ is strongly robust.

We now illustrate Definition 3.1.2 by means of two simple examples in the case of first order systems.

Example 3.1.6 (Strong robustness and pole placement) We consider pole placement design in the case of first order systems in $\mathcal{P}_1$. Let α denote the desired closed-loop pole, with $|\alpha| < 1$. Using the notation introduced in Chapter 2, $\mathcal{P}_1$ consists of the discrete-time systems $(a, b)^T \in \mathbb{R}^2$ described by

$$y(k+1) + ay(k) = bu(k). \tag{3.8}$$

If $(a, b)^T \in \mathcal{C}_1$, i.e., if $b \neq 0$, the feedback controller achieving pole placement in α associated with (3.8) leads to the control law

$$u(k) = \frac{a+\alpha}{b} y(k) \tag{3.9}$$

Hence, a set $\Omega \subset \mathcal{C}_1$ is strongly robust with respect to pole placement in α if and only if for any system $(a, b)^T \in \Omega$ and for any sequence of systems $\{(a(k), b(k))^T\}_{k \in \mathbb{N}} \subset \Omega$, the closed-loop system described by

$$y(k+1) = (-a + b\frac{a(k)+\alpha}{b(k)})y(k) \tag{3.10}$$

is asymptotically stable. Thus $\Omega \subset \mathcal{C}_1$ is strongly robust with respect to pole placement in α if and only if for any system $(a, b)^T \in \Omega$ and for any sequence $\{(a(k), b(k))^T\}_{k \in \mathbb{N}} \subset \Omega$, the absolute value of the time-varying closed-loop pole of the system (3.10) is smaller than 1 and stays bounded away from 1. Formally, $\Omega \subset \mathcal{P}_1$ is strongly robust with respect to pole placement in α if and only if for any system $(a, b)^T \in \Omega$, $b \neq 0$ and

$$\exists \epsilon \in]0,1[: |-a + b\frac{a(k)+\alpha}{b(k)}| < 1-\epsilon, \forall (a,b)^T \in \Omega, \forall \{(a(k), b(k))^T\}_{k \in \mathbb{N}} \subset \Omega. \tag{3.11}$$

■

Example 3.1.7 (Strong robustness and Linear Quadratic control) We consider Linear Quadratic (LQ) control design in the case of first order systems in $\mathcal{P}_1$. Suppose that the LQ control objective is to minimize the quadratic cost criterion

$$J = \sum_{k=0}^{\infty} y(k)^2 + ru(k)^2, \tag{3.12}$$

where the weight $r > 0$ is given. If $(a, b)^T \in \mathcal{C}_1$, the feedback control input u minimizing J given in (3.12) associated with (3.8) is given by [6]:

$$u(k) = f(a,b)y(k) = -\frac{bp(a,b)}{r} y(k), \tag{3.13}$$

where $p(a, b)$ is equal to the unique positive root p of the Algebraic Riccati Equation

$$-\frac{1}{r}b^2p^2 - 2ap + 1 = 0. \tag{3.14}$$

Hence the input sequence that minimizes the criterion given in (3.12) is

$$u(k) = [\frac{a}{b} - \frac{1}{rb}\sqrt{a^2r^2 + b^2r}]y(k). \tag{3.15}$$

It follows from the discussion in Example 3.1.6 that $\Omega \subset \mathcal{P}_1$ is strongly robust with respect to LQ control with quadratic cost (3.12) if and only if for any system $(a, b)^T \in \Omega$, $b \neq 0$ and there exists $\epsilon \in]0, 1[$ such that

$$|-a + b[\frac{a(k)}{b(k)} - \frac{1}{rb(k)}\sqrt{a^2(k)r^2 + b^2(k)r}] < 1-\epsilon, \forall (a, b)^T \in \Omega, \forall \{(a(k), b(k))^T\}_{k\in\mathbb{N}} \subset \Omega. \tag{3.16}$$

■

In Definition 3.1.2, the notion of asymptotic stability plays a crucial role but could theoretically be replaced by other stability notions. For instance, involving quadratic stability [111], the notion of *strongly quadratically robust sets of systems* is defined as follows.

Definition 3.1.8 (Strong quadratic robustness) *Let $\Omega \subset \mathcal{P}_n$ and suppose that the control objective is given and satisfies Assumption 2.1.13. Ω is strongly quadratically robust with respect to the given control objective if $\Omega \subset \mathcal{P}_n$ and for any system $\theta \in \Omega$ there exists a quadratic Lyapunov function for the time-varying system defined by* (3.7), *for any sequence of systems $\{\theta(k)\}_{k\in\mathbb{N}} \subset \Omega$. More precisely, Ω is strongly quadratically robust with respect to the given control objective if the two following statements hold:*

- *$\Omega \subset \mathcal{P}_n$;*
- *for any system $\theta \in \Omega$ there exists a matrix $K = K^T > 0$ in $\mathbb{R}^{(2n-1)\times(2n-1)}$ such that for any sequence of systems $\{\theta(k)\}_{k\in\mathbb{N}} \subset \Omega$, the following matrix inequality is satisfied:*

$$[A(\theta) + B(\theta)f(\theta(k))]^T K[A(\theta) + B(\theta)f(\theta(k))] - K + I < 0. \tag{3.17}$$

where $A(\theta)$, $B(\theta)$ and $f(\theta)$ are given by

$$A(\theta) = \begin{bmatrix} -a_{n-1} & \cdots & \cdots & -a_1 & -a_0 & b_{n-2} & \cdots & \cdots & b_1 & b_0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & & \vdots & \vdots & \vdots & & & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots & \vdots & & & \vdots & \vdots \\ \vdots & & & 1 & \vdots & \vdots & & & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & & & 0 & \vdots & 1 & & & \vdots & \vdots \\ \vdots & & & \vdots & \vdots & 0 & \ddots & & \vdots & \vdots \\ \vdots & & & \vdots & \vdots & \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 1 & 0 \end{bmatrix} \tag{3.18}$$

$$B(\theta) = \begin{bmatrix} b_{n-1} & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}^T \tag{3.19}$$

with

$$\theta = (a_{n-1}, \cdots, a_0, b_{n-1}, \cdots, b_0)^T \in \mathbb{R}^{2n} \tag{3.20}$$

and $f(\theta)$ is the controller defined in Assumption 2.1.13.

Example 3.1.9 (Strong quadratic robustness and pole placement) We now revisit Example 3.1.6 to illustrate the notion of quadratic strong robustness in the case of first order pole placement design. It follows from Definition 3.1.8 that a set $\Omega \subset \mathcal{C}_1$ is strongly quadratically robust with respect to pole placement in α if and only if for any system $(a,b)^T \in \Omega$ and for any sequence of systems $\{(a(k),b(k))^T\}_{k\in\mathbb{N}} \subset \Omega$, there exists a quadratic Lyapunov function for the time-varying closed-loop system described by (3.10). Thus $\Omega \subset \mathcal{C}_1$ is strongly quadratically robust with respect to pole placement in α if and only if for any system $(a,b)^T \in \Omega$ and for any sequence $\{(a(k),b(k))^T\}_{k\in\mathbb{N}} \subset \Omega$, there exists $K \in \mathbb{R}$, $K > 0$ such that

$$|-a+b\frac{a(k)+\alpha}{b(k)}|.K.|-a+b\frac{a(k)+\alpha}{b(k)}|-K+1 < 0, \forall (a,b)^T \in \Omega, \forall \{(a(k),b(k))^T\}_{k\in\mathbb{N}} \subset \Omega$$

equivalently if and only if there exists $K \in \mathbb{R}$, $K > 0$ such that

$$|-a+b\frac{a(k)+\alpha}{b(k)}|^2 < 1 - \frac{1}{K}, \forall (a,b)^T \in \Omega, \forall \{(a(k),b(k))^T\}_{k\in\mathbb{N}} \subset \Omega. \tag{3.21}$$

(3.21) is equivalent to (3.11), meaning that in the first order case, a set of systems is strongly quadratically robust with respect to pole placement in a stable pole α if and only if it is strongly robust with respect to pole placement in α. We easily check that this result also applies irrespective of the adopted control law. ■

It is shown in [111] that if the system (3.7) is quadratically stable then it is asymptotically stable for any sequence $\{\theta(k)\}_{k\in\mathbb{N}} \subset \Omega$. Thus if a set is strongly quadratically robust then it is also strongly robust. We see further in this chapter (see Section 3.3.3) that under some additional assumptions on the considered sets of systems, strongly quadratically robust sets of systems might be easier to characterize than strongly robust sets.

Now, it is clear that in order to be strongly robust, a set of systems in $\mathcal{C}_n$ is necessarily such that for any system in this set, the controller based on it stabilizes any other system, i.e., the property of strong robustness without time-variations of the involved controller is necessarily satisfied. This leads to a simplified notion of strong robustness, *the time-invariant strong robustness*.

Definition 3.1.10 (Time-invariant strong robustness) *Let $\Omega \subset \mathcal{P}_n$ and suppose that the control objective is given and satisfies Assumption 2.1.13. Ω is time-invariant strongly robust with respect to the given control objective if $\Omega \subset \mathcal{C}_n$ and for any systems $\theta, \theta' \in \Omega$, the system defined by*

$$\begin{aligned} y(k+1) &= \theta^T \phi(k) \\ u(k) &= f(\theta')x(k), \end{aligned} \tag{3.22}$$

where the vectors $\phi(k) \in \mathbb{R}^{2n}$ and $x(k) \in \mathbb{R}^{2n}$ are given in (3.4) *and* (3.6) *respectively, is asymptotically stable.*

Example 3.1.11 (Time-invariant strong robustness and pole placement) We consider first order pole placement design. Similar to Example 3.1.6, let α denote the desired closed-loop pole, with $|\alpha| < 1$. It follows from (3.11) that $\Omega \subset \mathcal{P}_1$ is time-invariant strongly robust with respect to pole placement in α if and only if $\Omega \subset \mathcal{C}_1$ and

$$|-a + b\frac{a' + \alpha}{b'}| < 1, \forall (a,b)^T, (a',b')^T \in \Omega. \tag{3.23}$$

Equation (3.23) has a simple interpretation in the Euclidean plane as illustrated in Figure 3.1. A set of systems is time-invariant strongly robust if it is completely contained within the interior of a parallelogram bounded by parallel lines going through the points $(1, 0)$ and $(\alpha, 0)$, as well as parallel lines through the points $(-1, 0)$ and $(\alpha, 0)$. The uncontrollable system $(\alpha, 0)$ is thus at the corner of these maximally strongly robust sets of systems. ■

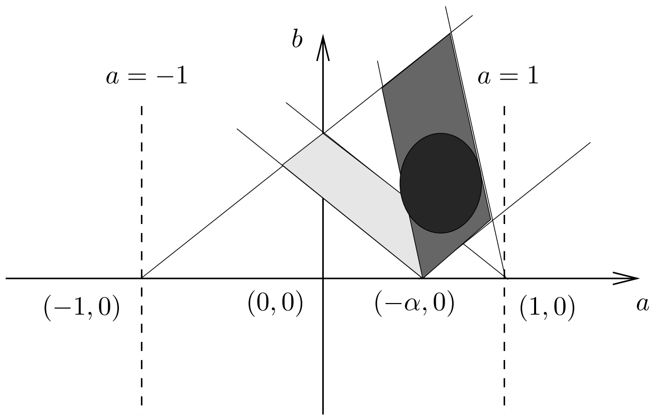


Figure 3.1: Time-invariant strong robustness with respect to pole-placement in α.

Remark 3.1.12 Using the geometrical construction in Example 3.1.11, we easily check that the unique unbounded set containing a given system (a^0, b^0) which is strongly robust with respect to pole placement in a given stable pole α_0 is the line going through the points (a_0, b_0) and $(-\alpha, 0)$. This line is the set of systems θ yielding a controller $f(\theta)$ which is exactly the controller $f(\theta^0)$.

Example 3.1.13 (Time-invariant strong robustness and LQ control) Let us investigate the case of time-invariant strong robustness for first order LQ control placement design. Similar to Example 3.1.7, we focus on LQ control with a quadratic cost given in (3.12). It follows from (3.16) that $\Omega \subset \mathcal{P}_1$ is time-invariant strongly robust with respect to the LQ control defined by (3.12) if and only if $\Omega \subset \mathcal{C}_1$ and

$$|-a + b[\frac{a'}{b'} - \frac{1}{rb'}\sqrt{a'^2r^2 + b'^2r}]| < 1, \forall (a,b)^T, \forall (a',b')^T \in \Omega. \tag{3.24}$$

Equation (3.24) can be geometrically interpretation in the Euclidean plane $(a, b)^T$ as illustrated in Figure 3.2 and Figure 3.3. Let us first suppose that $r = 1$. Let (a_0, b_0) be a system in $\mathbb{R}^2$. Let Σ^0 denote the set of systems in $\mathbb{R}^2$ defined by:

$$\Sigma^0 = \{(a,b)^T : |-a + b[\frac{a_0}{b_0} - \frac{1}{b_0}\sqrt{a_0^2 + b_0^2}]| < 1\}. \quad (3.25)$$

It can be easily shown that Σ^0 represents the region between the two parallel lines with direction $\vec{v} = (\sqrt{a_0^2 + b_0^2} - a_0, -b_0)$ and going through the points $(1,0)$ and $(-1,0)$ respectively. This region is represented in Figure 3.2. Hence, given a set $\Omega \subset \mathbb{R}^2$, Ω is time invariant strongly robust with respect to LQ control defined in (3.12) with $r = 1$ if and only if it completely belongs to the parallelogram bounded by the parallel lines going through the points $(1,0)$ and $(-1,0)$ with direction given by $\vec{v}_1 = (\sqrt{a_1^2 + b_1^2} - a_1, -b_1)$ as well as the parallel lines going through the points $(1,0)$ and $(-1,0)$ with direction given by $\vec{v}_2 = (\sqrt{a_2^2 + b_2^2} - a_2, -b_2)$ where (a_1, b_1) and (a_2, b_2) are the points where the minimum-volume cone with vertex $(0,0)$ containing Ω intersects Ω. This construction is illustrated in Figure 3.3.

Then, if $r \neq 1$, denote by Ω_r the set of systems obtained by multiplying the coordinate b of the systems in ω by $\frac{1}{\sqrt{r}}$ defined by:

$$\Omega_r = \{(a, \frac{b}{\sqrt{r}})^T \in \mathbb{R}^2 : (a,b)^T \in \Omega\}. \quad (3.26)$$

Then the following result follows from our previous discussion. A set $\Omega \subset \mathbb{R}^2$ is time invariant strongly robust with respect to LQ control defined in (3.12) with $r > 0$ if and only if the set Ω_r defined in (3.26) completely belongs to the parallelogram bounded by the parallel lines going through the points $(1,0)$ and $(-1,0)$ with direction given by $\vec{v}_1 = (\sqrt{a_1^2 + b_1^2} - a_1, -b_1)$ as well as the parallel lines going through the points $(1,0)$ and $(-1,0)$ with direction given by $\vec{v}_2 = (\sqrt{a_2^2 + b_2^2} - a_2, -b_2)$ where (a_1, b_1) and (a_2, b_2) are the points where the minimum-volume cone with vertex $(0,0)$ containing Ω_r intersects Ω_r. ■

Time-invariant strong robustness is a weaker notion than strong robustness, meaning that the former does not imply the latter. Indeed, already in the first order case ($n = 1$), time-variations of the controller do play a role in the stability of the closed-loop system. To prove this result, let us consider pole placement in α, $|\alpha| < 1$. Consider a set $\Omega \subset \mathcal{C}_1$ such that there exists a system $\theta^0 \in \Omega$ and a sequence of systems $\{\theta(k)\}_{k\in\mathbb{N}} \subset \Omega$ such that

$$\lim_{k\to\infty} b(k) = b^0 \text{ and } \lim_{k\to\infty} a(k) = 1 + a^0 - \alpha \quad (3.27)$$

and such that

$$|-a^0 + b^0\frac{\alpha + a(k)}{b(k)}| < 1, \forall k \in \mathbb{N}. \quad (3.28)$$

(3.28) implies that

$$\lim_{k\to\infty} |-a^0 + b^0\frac{\alpha + a(k)}{b(k)}| = 1, \quad (3.29)$$

thus (3.23) is satisfied, hence Ω is time-invariant strongly robust with respect to pole placement in α. However, (3.27) implies that (3.11) is not satisfied, hence Ω is not strongly robust with respect to pole placement in α. Nevertheless we have the following property.

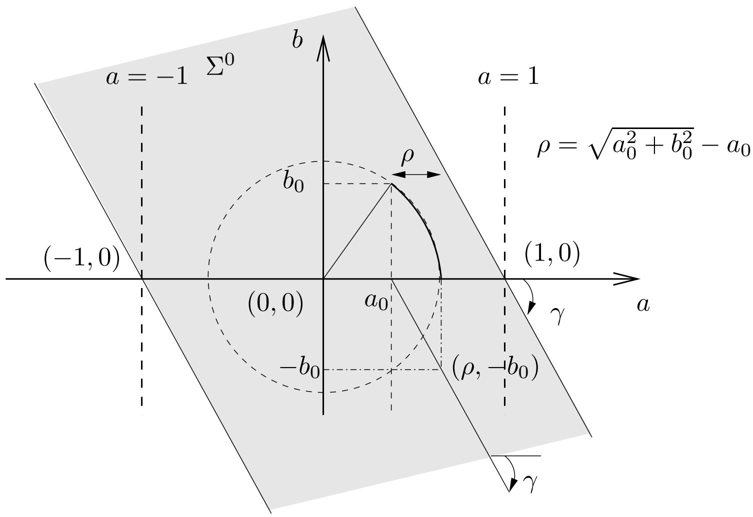


Figure 3.2: Time-invariant strongly robust set with respect to LQ control, r=1

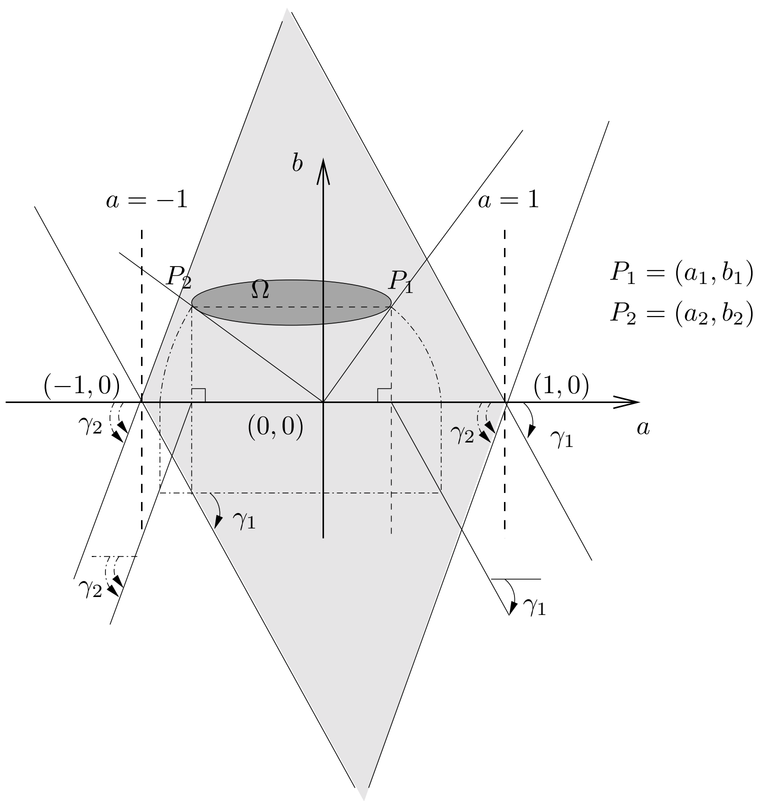


Figure 3.3: Time-invariant strongly robust set with respect to LQ control, r=1

Property 3.1.14 *Let $\Omega \subset \mathcal{C}_1$ be a compact set. Suppose that the control objective is given and satisfies Assumption 2.1.13. Then Ω is strongly robust (equivalently strongly quadratically robust) with respect to this control objective if and only if it is time-invariant strongly robust with respect to this control objective.*

Proof: Let $\Omega \subset \mathcal{C}_1$ be a compact set. Suppose that the control objective is given and satisfies Assumption 2.1.13. Suppose that Ω is time-invariant strongly robust but not strongly robust. Then, we have

$$|-a+bf(a',b')|<1, \forall (a,b)^T, \forall (a',b')^T \in \Omega, \tag{3.30}$$

and there exists a system $(a_0,b_0)^T \in \Omega$ and there exists a sequence of systems $\{(a(k),b(k))^T\}_{k\in\mathbb{N}} \subset \Omega$ such that

$$\lim_{k\to\infty} |-a^0+b^0 f(a(k),b(k))| = 1. \tag{3.31}$$

Now, due to the continuity of f (Assumption 2.1.13), the function defined by

$$g:(a,b)^T \in \Omega \to |-a^0+b^0 f(a,b)| \in [0,1[\tag{3.32}$$

is continuous. Now, it follows from the compactness of Ω that

$$\lim_{k\to\infty} |-a^0+b^0 f(a(k),b(k))| \in \{|-a+bf(a',b')| : (a,b)^T \in \Omega, (a',b')^T \in \Omega\} \tag{3.33}$$

thus (3.31) implies

$$1 \in \{|-a+bf(a',b')| : (a,b)^T \in \Omega, (a',b')^T \in \Omega\}. \tag{3.34}$$

However, (3.34) contradicts (3.30). Hence the assumption that Ω is not strongly robust is falsified. Therefore Ω is strongly robust. In Example 3.1.9, we have shown that strong robustness and strong quadratical robustness are equivalent in the first order case. ■

In general, in the higher order case, the three notions defined in Definition 3.1.2, Definition 3.1.10 and Definition 3.1.8 are not equivalent. Strong quadratic robustness implies strong robustness, which in turn implies time-invariant strong robustness.

Remark 3.1.15 Whether or not a given set is strongly robust depends on the control objective. This indicates a link between performance and uncertainty. If an uncertainty set is not strongly robust with respect to a particular control objective, it may be strongly robust with respect to another control objective. Information about the uncertainty set may be used to find the most suitable control objective. This is an important property in control design, normally lacking from classical adaptive control discussions.

Remark 3.1.15 suggests to revisit once again the notion of strong robustness and introduce the notion of weak strong robustness as follows.

Definition 3.1.16 (Weak strong robustness) *Let $\Omega \subset \mathcal{P}_n$ and let $\mathcal{F}$ be a class of control objectives that satisfy Assumption 2.1.13. Ω is weakly strongly robust (respectively weakly strongly quadratically robust, weakly time-invariant strongly robust) with respect to $\mathcal{F}$ if there exists a control objective in $\mathcal{F}$ with respect to which Ω is strongly robust (respectively strongly quadratically robust, time-invariant strongly robust).*

Example 3.1.17 (Weak strong robustness and pole placement) We revisit Example 3.1.6 so as to illustrate the notion of weak strong robustness. It follows from Definition 3.1.16 that a set $\Omega \subset \mathcal{P}_1$ is weakly strongly robust with respect to pole placement in a stable pole if and only if there exists a stable pole such that Ω is strongly robust with respect to pole placement in this pole, i.e., $\forall (a,b)^T \in \Omega$, $b \neq 0$ and

$$\exists \alpha \in]-1,1[, \exists \epsilon \in]0,1[: |-a+b\frac{a(k)+\alpha}{b(k)}| < 1-\epsilon, \forall (a,b)^T \in \Omega, \forall \{(a(k),b(k))^T\}_{k\in\mathbb{N}} \subset \Omega. \tag{3.35}$$

A geometrical interpretation of (3.35) is presented in Figure 3.4. We construct $\mathcal{T}_+$ as the line tangent to Ω going through $(1,0)$ on the right-hand side of Ω. We denote by $\mathcal{T}'_+$ the line parallel to $\mathcal{T}_+$ and tangent to Ω on the left-hand side of Ω. We denote by $\overline{\alpha}$ the intersection between $\mathcal{T}'_+$ and the a-axis. Similarly, we construct $\mathcal{T}_-$ as the line tangent to Ω going through $(-1,0)$ on the left-hand side of Ω. We denote by $\mathcal{T}'_-$ the line parallel to $\mathcal{T}_-$ and tangent to Ω on the right-hand side of Ω. We denote by $\underline{\alpha}$ the intersection between $\mathcal{T}'_-$ and the a-axis. It can easily be checked that the set Ω is weakly strongly robust (according to (3.35)) if and only if $-1 < \underline{\alpha} \leq \overline{\alpha} < 1$. More precisely, if $-1 < \underline{\alpha} \leq \overline{\alpha} < 1$, Ω is strongly robust with respect to pole placement in any stable pole $\alpha \in [\underline{\alpha}, \overline{\alpha}]$. If $\underline{\alpha} > \overline{\alpha}$ or if $[\underline{\alpha}, \overline{\alpha}] \cap]-1,1[= \emptyset$, then Ω is not weakly strongly robust with respect to any pole placement in a stable pole. ■

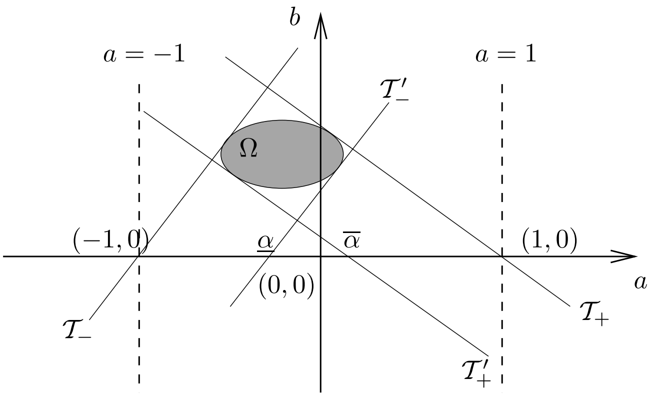


Figure 3.4: Weak strong robustness with respect to pole-placement.

In the remaining of this chapter, we adopt the following convention.

Convention 3.1.18 When the control objective is not specified, it is assumed to be fixed, a priori given, and it satisfies Assumption 2.1.13. The associated control law introduced in Assumption 2.1.13 will be denoted by f. Moreover, for any set $\Omega \subset \mathcal{C}_n$, we denote by $f(\Omega)$ the set of controllers associated with systems in Ω, defined by

$$f(\Omega) = \{f(\theta) : \theta \in \Omega\}. \tag{3.36}$$

3.1.2 Strong robustness radius

It follows from Assumption 2.1.13 that for any given system $\theta^0 \in \mathcal{C}_n$, the closed loop system $(\theta^0, f(\theta^0))$ defined by

$$\begin{aligned} y(k+1) &= (\theta^0)^T \phi(k) \\ u(k) &= f(\theta^0)x(k) \end{aligned} \tag{3.37}$$

is asymptotically stable. Otherwise stated, the set $\{\theta^0\}$ is strongly robust. Now if $\theta^0 \in \mathcal{P}_n \setminus \mathcal{C}_n$, Definition 3.1.2 implies that for any control objective satisfying Assumption 2.1.13, there does not exist any strongly robust set containing θ^0. To go further, we now examine the following issue: for a given system $\theta^0 \in \mathcal{C}_n$, what is the magnitude of the largest perturbation on θ^0 such that the set described by the perturbed system is strongly robust? This leads to the notion of *strong robustness radius*.

Strong robustness radius

The strong robustness radius around a given system in $\mathcal{P}_n$ is defined as follows.

Definition 3.1.19 (Strong robustness radius) *Let $\theta^0 \in \mathcal{P}_n$. We call strong robustness radius around θ^0 the radius $\rho^{\mathrm{SR}}(\theta^0)$ of the largest strongly robust ball of systems with center θ^0:*

$$\rho^{\mathrm{SR}}(\theta^0) = \max_{\Delta_0 \geq 0} \{\Delta_0 \in \mathbb{R}^{2n} : \{\theta = \theta^0 + \Delta\theta : \Delta\theta \in \mathbb{R}^{2n}, ||\Delta\theta|| \leq \Delta_0\} \textit{ is strongly robust}\}. \tag{3.38}$$

By convention, if $\theta^0 \in \mathcal{P}_n \setminus \mathcal{C}_n$, we use the notation: $\rho^{\mathrm{SR}}(\theta^0) = 0$.

Of course, a notion similar to strong robustness radius can be extended to lead to the notion of strong quadratic robustness radius around θ^0, denoted by $\rho^{\mathrm{QSR}}(\theta^0)$. Naturally, we have:

$$0 \leq \rho^{\mathrm{QSR}}(\theta^0) \leq \rho^{\mathrm{SR}}(\theta^0), \forall \theta^0 \in \mathcal{C}_n. \tag{3.39}$$

We now introduce the following notation.

Notation 3.1.20 *For a given system $\theta^0 \in \mathcal{P}_n$, we denote by $\mathcal{T}_{\theta^0}$ the largest set of systems in $\mathcal{C}_n$ containing θ^0 such that for any sequence of systems $\{\theta(k)\}_{k \in \mathbb{N}} \subset \mathcal{T}_{\theta^0}$, the time varying system defined by*

$$\begin{aligned} y(k+1) &= (\theta^0)^T \phi(k) \\ u(k) &= f(\theta(k))x(k), \end{aligned} \tag{3.40}$$

where the vectors $\phi(k) \in \mathbb{R}^{2n}$ and $x(k) \in \mathbb{R}^{2n}$ are given in (3.4) *and* (3.6) *respectively, is asymptotically stable. Moreover we denote by r_{θ^0} the radius of the largest sphere with center θ^0 that is contained in $\mathcal{T}_{\theta^0}$. By convention, if $\theta^0 \in \mathcal{P}_n \setminus \mathcal{C}_n$, then $\mathcal{T}_{\theta^0} = \emptyset$ and $r_{\theta^0} = 0$.*

For any $\theta^0 \in \mathcal{C}_n$, the set $\mathcal{T}_{\theta^0}$ defined in Notation 3.1.20 exists since $\{\theta^0\} \subset \mathcal{T}_{\theta^0}$. Any strongly robust set of systems in $\mathcal{C}_n$ containing θ^0 is a subset of $\mathcal{T}_{\theta^0}$. Hence we have:

$$r_{\theta^0} \geq \rho^{\mathrm{SR}}(\theta^0), \ \forall \theta^0 \in \mathcal{P}_n, \tag{3.41}$$

where $\rho^{\mathrm{SR}}(\theta^0)$ is defined in (3.38). We have the following result.

Theorem 3.1.21

$$\text{A set } \Omega \subset \mathcal{P}_n \text{ is strongly robust } \Leftrightarrow \Omega \subset \bigcap_{\theta \in \Omega} \mathcal{T}_\theta. \tag{3.42}$$

Proof: it directly follows from Definition 3.1.2 ■

Theorem 3.1.21 leads to the following corollary.

Corollary 3.1.22 *Let Ω be a non-empty ball of systems in $\mathcal{P}_n$ and let $r(\Omega)$ denote its radius. We have the following results.*

- **i.** *If $r(\Omega) \leq \frac{1}{2}\min_{\theta\in\Omega} r_\theta$ then Ω is strongly robust, where r_θ is defined in Notation 3.1.20.*
- **ii.** *If Ω is strongly robust, then $r(\Omega) \leq \frac{1}{2}\min_{\theta\in\partial(\Omega)} \overline{d}_\theta$, where $\partial(\Omega)$ is the boundary of Ω and $\overline{d}_\theta = \max_{\theta'\in\mathcal{T}_\theta} ||\theta' - \theta||$, $\forall\theta \in \mathcal{C}_n$, where $\mathcal{T}_\theta$ is defined in Notation 3.1.20.*

Proof:
i. Suppose Ω to be a non-empty ball of systems in $\mathcal{P}_n$ and suppose that its radius $r(\Omega)$ satisfies

$$r(\Omega) \leq \frac{1}{2}\min_{\theta\in\Omega} r_\theta. \tag{3.43}$$

Now, for any $\theta, \theta' \in \Omega$, we have: $||\theta - \theta'|| < 2r(\Omega)$. Hence, for any $\theta, \theta' \in \Omega$, we have:

$$||\theta - \theta'|| < \min_{\theta''\in\Omega} r_{\theta''} \leq r_\theta. \tag{3.44}$$

Thus for any $\theta, \theta' \in \Omega$, θ' belongs to the largest ball of systems with center θ and with radius r_θ, hence $\Omega \subset \mathcal{T}_\theta$, $\forall\theta \in \Omega$. Equivalently, we have: $\Omega \subset \bigcap_{\theta\in\Omega} \mathcal{T}_\theta$. It follows from Theorem 3.1.21 that Ω is strongly robust. Hence **i.**.

ii. Suppose Ω to be a strongly robust ball of systems in $\mathcal{P}_n$. Then, it follows from Theorem 3.1.21 that for any $\theta, \theta_i \in \Omega$, we have that $\theta_i \in \mathcal{T}_{\theta^0}$. Hence we have:

$$\forall\theta, \theta_i \in \Omega,\ ||\theta - \theta_i|| \leq \overline{d}_\theta = \max_{\theta'\in\mathcal{T}_\theta} ||\theta' - \theta||. \tag{3.45}$$

Therefore,

$$\forall\theta, \theta_i \in \Omega,\ \max_{\theta_i\in\Omega} ||\theta - \theta_i|| \leq \overline{d}_\theta. \tag{3.46}$$

Equivalently, we have

$$\forall\theta, \theta_i \in \Omega,\ \max_{\theta_i\in\partial(\Omega)} ||\theta - \theta_i|| \leq \overline{d}_\theta. \tag{3.47}$$

Since $\max_{\theta_i\in\partial(\Omega)} ||\theta - \theta_i|| = 2r(\Omega)$, (3.47) is equivalent to:

$$\forall\theta \in \partial(\Omega),\ r(\Omega) \leq \frac{1}{2}\overline{d}_\theta. \tag{3.48}$$

and therefore we have $r(\Omega) \leq \frac{1}{2}\min_{\theta\in\partial(\Omega)} \overline{d}_\theta$, which concludes the proof of **ii.** ■

Time-invariant strong robustness radius

The notion of strong robustness radius around a given system in $\mathcal{P}_n$ can be restricted to the time-invariant case. In this respect, the time-invariant strong robustness radius [25] around a given system in $\mathcal{C}_n$ is defined as follows.

Definition 3.1.23 (Time-invariant strong robustness radius) *Let $\theta^0 \in \mathcal{P}_n$. We call time-invariant strong robustness radius around θ^0 the radius $\rho^{\mathrm{TISR}}(\theta^0)$ of the largest time-invariant strongly robust ball of systems with center θ^0:*

$$\rho^{\mathrm{TISR}}(\theta^0) = \max_{\Delta_0 \geq 0}\{\Delta_0 : \{\theta = \theta^0 + \Delta : \Delta \in \mathbb{R}^{2n}, ||\Delta|| \leq \Delta_0\} \text{ is time-invariant strongly robust}\}.$$

By convention, if $\theta^0 \in \mathcal{P}_n \setminus \mathcal{C}_n$, we will note: $\rho^{\mathrm{TISR}}(\theta^0) = 0$.

Naturally, we have:

$$\rho^{\mathrm{SR}}(\theta^0) \leq \rho^{\mathrm{TISR}}(\theta^0), \forall \theta^0 \in \mathcal{P}_n, \tag{3.49}$$

where $\rho^{\mathrm{SR}}(\theta^0)$ is defined in (3.38) We now introduce the set of stabilizing controllers.

Definition 3.1.24 (Set of stabilizing controllers for systems in $\mathcal{C}_n$) *Suppose that the control objective is fixed and satisfies Assumption 2.1.13. Given a system $\theta \in \mathcal{C}_n$, we denote by $\mathcal{S}_\theta$ the set of controllers that stabilize θ defined by*

$$\mathcal{S}_\theta = \{\varphi \in f(\mathcal{C}_n) : A(\theta) + B(\theta)\varphi \text{ is Schur stable}\}, \tag{3.50}$$

where $A(\theta)$ and $B(\theta)$ are given in (3.18) *and* (3.19) *respectively. By convention, if $\theta \in \mathcal{P}_n \setminus \mathcal{C}_n$, then $\mathcal{S}_\theta = \emptyset$.*

We now introduce the following notation.

Notation 3.1.25 *$\forall \theta \in \mathcal{C}_n$, let $r^{\mathrm{TI}}_{f(\theta)}$ denote the radius of the largest open ball in $f(\mathcal{C}_n)$ centered about $f(\theta)$ contained in $\mathcal{S}_\theta$:*

$$r^{\mathrm{TI}}_{f(\theta)} = \sup\{\epsilon \geq 0 : \forall \varphi \in f(\mathcal{C}_n), ||\varphi - f(\theta)||_{\mathbb{R}} \leq \epsilon \Rightarrow \varphi \in \mathcal{S}_\theta\}. \tag{3.51}$$

We have the following theorem.

Theorem 3.1.26

$$\text{A set } \Omega \subset \mathcal{P}_n \text{ is time-invariant strongly robust} \Leftrightarrow \Omega \subset \mathcal{C}_n \text{ and } f(\Omega) \subset \bigcap_{\theta \in \Omega} \mathcal{S}_\theta. \tag{3.52}$$

Proof: from Definition 3.1.10, $\Omega \subset \mathcal{P}_n$ is time-invariant strongly robust if and only if $\Omega \subset \mathcal{C}_n$ and $\forall \theta, \theta' \in \Omega$, $f(\theta)$ stabilizes θ'. This is equivalent to say that $\forall \theta, \theta' \in \Omega$, $f(\theta') \subset \mathcal{S}_\theta$, equivalently $\forall \theta \in \Omega$, $f(\Omega) \subset \mathcal{S}_\theta$. ∎

Theorem 3.1.26 leads to the following corollary.

Corollary 3.1.27 *Let Ω be a set of systems in $\mathcal{C}_n$. And let $r(f(\Omega))$ denote the radius of the largest sphere of controllers in $f(\mathcal{C}_n)$ contained in $f(\Omega)$. We have the following results.*

i. *If* $r(f(\Omega)) \leq \frac{1}{2}\min_{\theta\in\Omega} r^{\mathrm{TI}}_{f(\theta)}$ *then* Ω *is time-invariant strongly robust, where* r^{TI}_{θ} *is defined in Notation 3.1.25.*

ii. *If* Ω *is time-invariant strongly robust, then* $r(f(\Omega)) \leq \frac{1}{2}\min_{f\in\partial(f(\Omega))} \overline{d}^{\mathrm{TI}}_{\theta}$, *where* $\partial(f(\Omega))$ *is the boundary of* Ω *and* $\overline{d}^{\mathrm{TI}}_{\theta} = \max_{\varphi\in\mathcal{S}_\theta} ||\varphi - f(\theta)||$, $\forall\theta \in \mathcal{C}_n$, *where* $\mathcal{S}_\theta$ *is defined in Definition 3.1.24.*

Proof: the proof is similar to the proof of Corollary 3.1.22. ■

3.2 Strong robustness measures

Although Theorem 3.1.21 and Theorem 3.1.26 (respectively Corollary 3.1.22 and Corollary 3.1.27) give some theoretical tests to check whether a given set of systems in $\mathcal{C}_n$ is strongly robust (respectively time invariant strongly robust) or not, the involved quantities r_θ, $\overline{d}_\theta$ (respectively $r^{\mathrm{TI}}_{f(\theta)}$, $\overline{d}^{\mathrm{TI}}_{\theta}$) and their min/max values are not easy to compute.
Over the last decade the analysis of the classical robustness notion and the issue of robustness measures for linear systems under complex and real perturbation received a good deal of attention [37], [51], [52]. In this context, the concept of structured stability radius [52] has been defined in the case of real or complex perturbations and an algorithm is given in [50] for the computation of this radius in the complex case. In this section, our aim is to exploit some of these results so as to express strong robustness measures using the well-studied notion of structured stability radius.

3.2.1 Structured stability radii and related notions

Real structured stability radius

The real stability radius of a Schur matrix under structured perturbation is defined as follows [51].

Definition 3.2.1 (Real structured stability radius) *Let* $M \in \mathbb{R}^{N\times N}$ *denote a strictly Schur stable matrix. The real stability radius of* M *with respect to the perturbation structure* $(D, E) \in \mathbb{R}^{N\times 1} \times \mathbb{R}^{1\times N}$, *is defined by [51]:*

$$r_{\mathbb{R}}(M, D, E) = \inf\{||\Delta||_{\mathbb{R}} : D \in \mathbb{R}^{1\times N}, M + D\Delta E \text{ is not Schur stable}\}, \tag{3.53}$$

where $||.||_{\mathbb{R}}$ *denotes the matrix norm in* $\mathbb{R}^{1\times N}$.

We have the following result.

Result 3.2.2 $\forall\theta \in \mathcal{C}_n$, $r^{\mathrm{TI}}_{f(\theta)} = r_{\mathbb{R}}(A(\theta) + B(\theta)f(\theta), B(\theta), I_{2n-1})$ *where* I_{2n-1} *is the unit matrix in* $\mathbb{R}^{(2n-1)\times(2n-1)}$ *and* $r_{\mathbb{R}}(A(\theta)+B(\theta)f(\theta), B(\theta), I_{2n-1})$ *is given in Definition 3.2.1 and* $r^{\mathrm{TI}}_{f(\theta)}$ *in notation 3.1.25.*

Proof: $r^{\mathrm{TI}}_{f(\theta)}$ can be expressed as follows

$$r^{\mathrm{TI}}_{f(\theta)} = \sup_{\varphi \in f(\mathcal{C}_n)} \{||\varphi - f(\theta)||_{\mathbb{R}} : A(\theta) + B(\theta)\varphi \text{ is strictly Schur stable}\}, \text{ or:}$$

$$r^{\mathrm{TI}}_{f(\theta)} = \inf_{\varphi \in f(\mathcal{C}_n)} \{||\varphi - f(\theta)||_{\mathbb{R}} : A(\theta) + B(\theta)\varphi \text{ is not Schur stable }\}, \text{i.e.,}$$

$$r^{\mathrm{TI}}_{f(\theta)} = \inf\{||\varphi - f(\theta)||_{\mathbb{R}} : \varphi \in \mathbb{R}^{2n-1}, A(\theta) + B(\theta)f(\theta) + B(\theta)(\varphi - f(\theta)) \text{ is not Schur stable}\}.$$

Therefore, $r^{\mathrm{TI}}_{f(\theta)} = r_{\mathbb{R}}(A(\theta) + B(\theta)f(\theta), B(\theta), I_{2n-1})$. ■

Complex structured stability radius

The complex stability radius of a Schur matrix under structured perturbation is defined as follows [51].

Definition 3.2.3 (Complex structured stability radius) *Let $M \in \mathbb{R}^{(2n-1)\times(2n-1)}$ denote a strictly Schur stable matrix. The complex stability radius of M with respect to the perturbation structure $(D, E) \in \mathbb{R}^{(2n-1)\times 1} \times \mathbb{R}^{1\times(2n-1)}$ is defined by [51]:*

$$r_{\mathbb{C}}(M, D, E) = \inf\{||\Delta||_{\mathbb{C}} : D \in \mathbb{C}^{1\times(2n-1)}, M + D\Delta E \text{ is not Schur stable}\}, \quad (3.54)$$

where $||.||_{\mathbb{C}}$ denotes the matrix norm in $\mathbb{C}^{1\times(2n-1)}$.

Remark 3.2.4 It is shown in [51] that the complex stability radius defined in Definition 3.2.3 does not change if the perturbation class is extended from static linear to the wider class of time-varying perturbations, whereas the real stability radius defined in Definition 3.2.1 depends on the specific perturbations class considered.

Property 3.2.5 *For all $\theta^0 \in \mathcal{C}_n$, $r_{\mathbb{C}}(A(\theta^0) + B(\theta^0)f(\theta^0), B(\theta^0), I_{2n-1}) > 0$*

Proof: the proof of Property directly follows from Definition 3.2.3. Indeed consider a system $\theta^0 \in \mathcal{C}_n$. Thus from Assumption 2.1.13, we have that

$$A(\theta^0) + B(\theta^0)f(\theta^0) \text{ is strictly Schur stable.} \quad (3.55)$$

Now suppose that $r_{\mathbb{C}}(A(\theta^0) + B(\theta^0)f(\theta^0), B(\theta^0), I_{2n-1}) = 0$. Using Definition 3.2.3, this implies that $A(\theta^0) + B(\theta^0)f(\theta^0) + B(\theta^0)0_{2n-1}$ is not Schur stable, denoting by 0_{2n-1} the zero matrix in $\mathbb{R}^{(2n-1)\times 1}$. This result contradicts (3.55). This concludes the proof of Property 3.2.5. ■

3.2.2 Structured stability radii and strong robustness

We now exploit the results in Section 3.2.1 to establish strong robustness measures. We first show how real structured stability radius and time-invariant strong robustness are connected.

Real structured stability radius and time-invariant strong robustness

From Theorem 3.1.26 we obtain the following result.

Corollary 3.2.6 *For any set $\Omega \subset \mathcal{C}_n$, if $\forall\theta, \theta' \in \Omega$,*

$$||f(\theta) - f(\theta')||_{\mathbb{R}} \leq r_{\mathbb{R}}(A(\theta) + b(\theta)f(\theta), B(\theta), I_{2n-1}), \tag{3.56}$$

then Ω is time-invariant strongly robust.

Proof: for a given set $\Omega \in \mathcal{C}_n$, suppose that (3.56) holds. Therefore, $\forall\theta, \theta' \in \Omega$,

$$||f(\theta') - f(\theta)||_{\mathbb{R}} \leq r_{\mathbb{R}}(A(\theta) + B(\theta)f(\theta), B(\theta), I_{2n-1}) \leq r^{\mathrm{TI}}_{f(\theta)}$$

hence $\forall\theta, \theta' \in \Omega, f(\Omega) \in \mathcal{S}_\theta$. Theorem 3.1.26 implies that S is time-invariant strongly robust. ■

Complex structured stability radius and strong robustness

We now show how the notions of complex structured stability radius and strong robustness are connected. We first introduce the following definition.

Definition 3.2.7 *For any system $\theta \in \mathcal{C}_n$, we denote by $\mathcal{B}(\theta)$ the ball of matrices in $\mathbb{R}^{1\times(2n-1)}$ centered in $f(\theta)$ with radius the complex stability radius $r_{\mathbb{C}}(A(\theta)+B(\theta)f(\theta), B(\theta), I_{2n-1})$, where I_{2n-1} is the unit matrix in $\mathbb{R}^{(2n-1)\times(2n-1)}$. More precisely, denoting by $f(\mathcal{C}_n)$ the set of controllers associated with systems in $\mathcal{C}_n$, we have:*

$$\mathcal{B}(\theta) = \{\varphi \in f(\mathcal{C}_n) : ||f(\theta)) - \varphi||_{\mathbb{R}} \leq r_{\mathbb{C}}(A(\theta) + B(\theta)f(\theta), B(\theta), I_{2n-1})\} \tag{3.57}$$

We have the following result:

Theorem 3.2.8 *For a given set $\Omega \subset \mathcal{C}_n$, if $f(\Omega) \subset \bigcap_{\theta\in\Omega} \mathcal{B}(\theta)$, then S is strongly robust, where $\mathcal{B}(\theta)$ is defined in* (3.57).

Proof: suppose $\Omega \subset \mathcal{C}_n$ to be such that $f(\Omega) \subset \bigcap_{\theta\in\Omega} \mathcal{B}(\theta)$. Then $\forall\theta \in \Omega$, and for all sequence $\{\theta(k)\}_{k\in\mathbb{N}} \subset \Omega$, we have

$$||f(\theta(k)) - f(\theta)||_{\mathbb{R}} \leq r_{\mathbb{C}}(A(\theta) + B(\theta)f(A(\theta), B(\theta)), B(\theta), I_{2n-1}). \tag{3.58}$$

Therefore, using Remark 3.2.4, $\forall\theta \in \Omega$, and for all sequence $\{\theta(k)\}_{k\in\mathbb{N}} \subset \Omega$, the closed-loop time-varying system with system matrix

$$A(\theta) + B(\theta)f(\theta) + B(\theta)(f(\theta(k)) - f(\theta)) = A(\theta) + B(\theta)f(\theta(k)) \tag{3.59}$$

is asymptotically stable, meaning that the time varying system defined by (3.7) is asymptotically stable, for all $\theta \in \mathcal{C}_n$. Hence it follows from Definition 3.1.2 that Ω is strongly robust. ■

Then, Theorem 3.2.8 yields the following result.

Theorem 3.2.9 *For any set $\Omega \subset \mathcal{C}_n$, if $\forall \theta, \theta' \in \Omega$,*

$$||f(\theta) - f(\theta')||_{\mathbb{R}} \leq r_{\mathbb{C}}(A(\theta) + B(\theta)f(\theta), B(\theta), I_{2n-1}), \tag{3.60}$$

then S is strongly robust.

Proof: suppose that (3.60) holds for a given set $\Omega \subset \mathcal{C}_n$. Then $\forall \theta, \theta' \in \Omega$, $f(\theta') \in \mathcal{B}(\theta)$. Hence, from Theorem 3.2.8, Ω is strongly robust. ■

Theorem 3.2.9 yields the following corollary.

Corollary 3.2.10 *Let Ω denote a subset of $\mathcal{C}_n$ and suppose that there exists a ball Σ of controllers in the set of controllers associated with systems in Ω such that*

$$f(\Omega) \subset \Sigma, \text{ i.e.}, \forall \theta \in \Omega, f(\theta) \in \Sigma. \tag{3.61}$$

Denoting by $r(\Sigma)$ the radius of Σ, if

$$r(\Sigma) \leq \frac{1}{2} \min_{\theta \in \Omega} r_{\mathbb{C}}(A(\theta) + B(\theta)f(\theta), B(\theta), I_{2n-1}), \tag{3.62}$$

then Ω is strongly robust.

Proof: the proof of Corollary is similar to the proof of **i.** in Theorem 3.1.21. Let $\Omega \subset \mathcal{C}_n$ and suppose that there exists a ball Σ in $f(\mathcal{C}_n$ such that (3.61) holds. Suppose that (3.62) is satisfied. Then, for any controllers φ, φ' in Σ, we have that

$$||\varphi - \varphi'||_{\mathbb{R}} \leq 2r(\Sigma) \leq \min \theta \in \Sigma r_{\mathbb{C}}(A(\theta) + B(\theta)f(\theta), B(\theta), I_{2n-1}). \tag{3.63}$$

From (3.61), $\forall \theta \in \Omega$, we have that $f(\theta) \in \Sigma$. Hence (3.63) implies that

$$||f(\theta) - f(\theta')||_{\mathbb{R}} \leq r_{\mathbb{C}}(A(\theta) + B(\theta)f(\theta), B(\theta), I_{2n-1}), \forall \theta, \theta' \in \Omega. \tag{3.64}$$

Hence (3.60) is satisfied. Corollary 3.2.10 hence follows from Theorem 3.2.9. ■

Remark 3.2.11 Theorem 3.2.9 has the great advantage that the problem of checking if a given set of systems in $\mathcal{C}_n$ is strongly robust (which a priori involves time-varying controllers) is reduced to a test involving time-invariant controllers only. Therefore, the characterization of strongly robust sets has been significantly simplified.

3.2.3 Existence of non-trivial strongly robust sets of systems

Before going further in the characterization of strongly robust sets of systems, we now focus on the issue of existence of non-trivial strongly robust sets, i.e., strongly robust sets of systems that are not reduced to a single point. We have the following result:

Theorem 3.2.12 (Existence of strongly robust open sets of systems in $\mathcal{C}_n$) *Around any system $\theta^0 \in \mathcal{C}_n$ there exists an open strongly robust neighborhood of systems in $\mathcal{C}_n$.*

Proof: let $\theta^0 \in \mathcal{C}_n$. Let us introduce the function:

$$\Gamma : \epsilon \in [0, \infty[\to \Gamma(\epsilon) = \min_{\theta:||\theta-\theta^0||\leq\epsilon} r_{\mathbb{C}}(A(\theta) + B(\theta)f(\theta), B(\theta), I_{2n-1}), \tag{3.65}$$

where $r_{\mathbb{C}}(A(\theta) + B(\theta)f(\theta), B(\theta), I_{2n-1})$ is defined in Definition 3.2.3.
It follows from the continuity of the complex stability radius [51] that Γ is continuous. Now, Property 3.2.5 implies that $r_{\mathbb{C}}(A(\theta^0) + B(\theta^0)f(\theta^0), B(\theta^0), I_{2n-1}) > 0$. Moreover, we have that for any system θ which belongs to $\mathcal{P}_n\ \mathcal{C}_n$, $r_{\mathbb{C}}(A(\theta) + B(\theta)f(\theta), B(\theta), I_{2n-1}) = 0$. Hence, we have

$$\Gamma(0) = r_{\mathbb{C}}(A(\theta^0) + B(\theta^0)f(\theta^0), B(\theta^0), I_{2n-1}) > 0 \text{ and } \lim_{x\to\infty} \Gamma(x) = 0, \tag{3.66}$$

It follows from the continuity of Γ and (3.66) that there exists $x_0 > 0$ such that:

$$\begin{aligned} x_0 &\leq \frac{\Gamma(0)}{4} = \frac{r_{\mathbb{C}}(A(\theta^0) + B(\theta^0)f(\theta^0), B(\theta^0), I_{2n-1})}{4} \\ \Gamma(x_0) &\geq \frac{\Gamma(0)}{2} = \frac{r_{\mathbb{C}}(A(\theta^0) + B(\theta^0)f(\theta^0), B(\theta^0), I_{2n-1})}{2}. \end{aligned} \tag{3.67}$$

Now, let B_0 denote the open ball of controllers in $f(\mathcal{C}_n)$ with radius x_0 and center $f(\theta^0)$. It follows from (3.67) that the radius $r(B_0)$ of B_0 is such that:

$$\begin{aligned} r(B_0) = x_0 &\leq \frac{r_{\mathbb{C}}(A(\theta^0) + B(\theta^0)f(\theta^0), B(\theta^0), I_{2n-1})}{4} \\ &\leq \frac{1}{2}\Gamma(x_0) \\ &= \frac{1}{2} \min_{\theta:||\theta-\theta^0||\leq x_0} r_{\mathbb{C}}(A(\theta) + B(\theta)f(\theta), B(\theta), I_{2n-1}), \end{aligned} \tag{3.68}$$

Now, using continuity of the map f on $\mathcal{C}_n$, we have that: $\exists\epsilon > 0$ such that if $||\theta - \theta'||_{\mathbb{R}} \leq \epsilon$, for $\theta, \theta' \in \mathcal{C}_n$, then $||f(\theta) - f(\theta')||_{\mathbb{R}} \leq r(B_0)$. Hence there exists an open ball Ω^0 of systems in $\mathcal{C}_n$ with center θ^0 and radius $\min(\epsilon, x_0) > 0$ such that $f(\Omega^0) \subset B_0$ and (3.68) implies that

$$r(B_0) \leq \frac{1}{2} \min_{\theta\in\Omega^0} r_{\mathbb{C}}(A(\theta) + B(\theta)f(\theta), B(\theta), I_{2n-1}). \tag{3.69}$$

Finally, according to Corollary 3.2.10, (3.69) implies that Ω^0 is strongly robust. This concludes the proof of Theorem 3.2.12. ■

Remark 3.2.13 In the proof of Theorem 3.2.12, the Assumption that the map f is continuous plays a crucial role. This motivates the continuity assumption in Assumption 2.1.13.

3.3 Testing strong robustness

The purpose in this section is to establish some tests that allow to check whether a given set in $\mathcal{C}_n$ enjoys the stability notions previously introduced in the previous section.

3.3.1 Testing controllability

A necessary condition that a set of systems in $\mathcal{P}_n$ has to satisfy in order to be strongly robust is to be contained in the set of controllable systems. In this subsection we are concerned with the following problem: given a set of systems $\Omega \subset \mathcal{P}_n$, construct a sufficient test to check whether Ω is a subset of $\mathcal{C}_n$ or not. As a first approach to this problem, we now go back to the basic properties of controllable linear systems.

Property 3.3.1 *A system $\theta \in \mathcal{P}_n$ is controllable if and only if any of the following statements holds:*

i. $A(\theta)$, $B(\theta)$ *given in* (3.18) *and* (3.19) *satisfy [93]:*

$$\mathrm{rank}([A(\theta) - \lambda I_{2n-1} B(\theta)]) = 2n-1, \forall \lambda \in \mathbb{C}; \tag{3.70}$$

ii. $A(\theta)$, $B(\theta)$ *given in* (3.18) *and* (3.19)*satisfy*

$$\min_{\lambda \in \mathbb{C}} \underline{\sigma}([A(\theta) - \lambda I_{2n-1}\ B(\theta)]) > 0, \tag{3.71}$$

where $\underline{\sigma}([A(\theta) - \lambda I_{2n-1}\ B(\theta)])$ *denotes the smallest singular value of the Hautus matrix* $([A(\theta) - \lambda I_{2n-1} B(\theta)])$ *[38];*

iii. $\mathrm{rank}(Sylv(A(\theta), B(\theta))) = 2n-1$ *[93], where* $Sylv(A(\theta), B(\theta))$ *denotes the Sylvester matrix given by:*

$$Sylv(A(\theta), B(\theta)) = \begin{bmatrix} a_0 & a_1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & a_0 & \cdots & a_{n-1} & 1 & \ddots & \vdots \\ \vdots & & \ddots & & & \ddots & 0 \\ 0 & \cdots & 0 & a_0 & a_1 & \cdots & 1 \\ \vdots & & & b_0 & b_1 & \cdots & b_{n-1} \\ \vdots & & b_0 & b_1 & \cdots & b_{n-1} & 0 \\ 0 & \iddots & \iddots & & \iddots & \ddots & \\ b_0 & b_1 & \cdots & b_{n-1} & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{(2n-1)\times(2n-1)}. \tag{3.72}$$

From these considerations, we derive the following result.

Property 3.3.2 *A set $\Omega \subset \mathcal{P}_n$ is a subset of $\mathcal{C}_n$ if and only if any of the following statements holds:*

i'. $A(\theta), B(\theta)$ *given in* (3.18) *and* (3.19) *satisfy*

$$\mathrm{rank}([A(\theta) - \lambda I_{2n-1} B(\theta)]) = 2n-1, \forall \lambda \in \mathbb{C}, \forall \theta \in \Omega; \tag{3.73}$$

ii'. $A(\theta), B(\theta)$ *given in* (3.18) *and* (3.19) *satisfy*

$$\min_{\lambda \in \mathbb{C}} \underline{\sigma}([A(\theta) - \lambda I_{2n-1}\ B(\theta)]) > 0, \forall \theta \in \Omega, \tag{3.74}$$

where $\underline{\sigma}([A(\theta) - \lambda I_{2n-1}\ B(\theta)])$ *denotes the smallest singular value of the Hautus matrix* $([A(\theta) - \lambda I_{2n-1} B(\theta)])$ *;*

iii'. The Sylvester matrix $Sylv(A(\theta), B(\theta))$ given in (3.72) *has full-row rank* $\forall\theta \in \Omega$.

Unfortunately, the characterization of controllability given in i'. is difficult to implement in finite precision [88]; indeed it is even not clear how to numerically verify whether a given system θ is controllable through (3.70). Hence checking whether a set is a subset of the set of controllable systems using i'. is not thinkable. Then, with respect to ii'., it has been shown [38] that

$$\underline{\sigma}([A(\theta) - \lambda I_{2n-1}\ B(\theta)]) = \min_{\Delta_A, \Delta_B} \{||\Delta_A, \Delta_B|| : \\ \text{rank}(Sylv(A(\theta) + \Delta_A, B(\theta) + \Delta_B)) \neq 2n - 1\}. \quad (3.75)$$

where $(\Delta_A, \Delta_B) \in \mathbb{R}^{(2n-1)\times(2n-1)} \times \mathbb{R}^{2n-1}$. In other words, (3.75) means that $\underline{\sigma}([A(\theta) - \lambda I_{2n-1}\ B(\theta)])$ is nothing but the distance from the pair of matrices $A(\theta), B(\theta)$ to the set of uncontrollable pairs of matrices in $\mathbb{R}^{(2n-1)\times(2n-1)} \times \mathbb{R}^{2n-1}$. This leads to the following result:

Theorem 3.3.3 *Let Ω denote a set of systems in $\mathcal{P}_n$ and let θ^* be any element in Ω. If* (3.74) *holds for $\theta = \theta^*$, and if*

$$||(A(\theta), B(\theta)) - (A(\theta^*), B(\theta^*))|| \leq \sigma_{\min}([A(\theta^*) - \lambda I_{2n-1}\ B(\theta^*)]), \forall\theta \in \Omega, \quad (3.76)$$

then $\Omega \subset \mathcal{C}_n$.

Hence, Theorem 3.3.3 may provide a method to check whether a set Ω is subset of $\mathcal{C}_n$ or not, choosing θ^* to be for instance the center of a ball of systems outer-bounding Ω . However, the function to be minimized in (3.74) is not convex and may have as many as $2n - 1$ or more local minima. Moreover it is not clear just how many local minima they are for a given system θ [24]. Many algorithms have been proposed in the literature to compute local minima of the function

$$\lambda \in \mathbb{C} \mapsto \text{Unc}(\theta) = \underline{\sigma}([A(\theta) - \lambda I_{2n-1}\ B(\theta)]), \quad (3.77)$$

for a given system $\theta \in \mathcal{P}_n$ ([20], [24], [113]), but have no guarantee of finding $\text{Unc}(\theta)$ with any accuracy, since $\text{Unc}(\theta)$ is the global minimum. Furthermore, methods that search for the global minimum ([24], [47], [36]) sometimes offer this guarantee but require a computation time that is inversely proportional to $(\text{Unc}(\theta))^2$, prohibitively large for nearly uncontrollable systems. For this reason, numerically tractable methods for estimating $\text{Unc}(\theta)$ for a given system θ can be found in [44]. On one hand, these algorithms require much smaller computation times, but on the other hand, the author shows that if $\text{Unc}(\theta)$ is very tiny, then its estimates by the proposed algorithms in finite precision could be much larger than the exact value. Further in Chapter 5, we will suppose that the true system to be controlled is unknown, hence we have no information about its controllability level. This implies that the case where the system is close to uncontrollability will not a priori be neglected in our framework.
Therefore, for our purposes, none of these reported methods based on i'. and ii'. is completely satisfactory. Alternatively, let us now exploit iii'. to derive a sufficient test to check whether a bounded set $\Omega \subset \mathcal{P}_n$ is subset of $\mathcal{C}_n$ or not. First, we have the following theorem [53].

Theorem 3.3.4 (Interval matrix and non-singularity) *Let* $\boldsymbol{S} = \hat{S} + [-\Delta, \Delta]$ *be an interval matrix in* $\mathbb{R}^{N\times N}$*, i.e.,*

$$S^1 \in \boldsymbol{S} \Leftrightarrow S_{i,j} - \Delta_{i,j} \leq S^1_{i,j} \leq S_{i,j} + \Delta_{i,j}, \forall i, j \leq N, \tag{3.78}$$

where, $\forall M \in \mathbb{R}^{N\times N}$*,* $M_{i,j}$ *denotes the entry of* M *in the* i*th row and* j*th column. For all* $M \in \mathbb{R}^{N\times N}$*, let* $\underline{\sigma}(M), \overline{\sigma}(M)$ *denote the smallest and the largest singular values of* M *respectively. Then* $\overline{\sigma}(\Delta) < \underline{\sigma}(\hat{S})$ *implies that* $\boldsymbol{S}$ *is non-singular.*

Hence, suppose that $\Omega \subset \mathcal{P}_n$ is a bounded set of systems. Recall that any element $\theta \in \mathcal{P}_n$ is of the form:

$$\theta = (a_{n-1}, \cdots, a_0, b_{n-1}, \cdots, b_0)^T. \tag{3.79}$$

Since Ω is bounded, we can enclose Ω in a symmetric outer-bounding polytopic set $\tilde{\Omega}$ of systems in $\mathcal{P}_n$, defined by known parameters $\{a_i^*, \Delta a_i, b_i^*, \Delta b_i\}_{i=0,\cdots,n-1}$ in $\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+$ such that

$$\tilde{\Omega} = \{\theta : a_i^* - \Delta a_i \leq a_i \leq a_i^* + \Delta a_i, b_i^* - \Delta b_i \leq b_i \leq b_i^* + \Delta b_i\}. \tag{3.80}$$

Now, for any element θ in $\tilde{\Omega}$ we can form the Sylvester $S(\theta)$ matrix given in (3.72). We have that

$$S(\theta) \in S^* + [-\Delta, \Delta], \forall \theta \in \tilde{\Omega}, \tag{3.81}$$

where

$$S^* = S(\theta^*) = S(a_{n-1}^*, \cdots, a_0^*, b_{n-1}^*, \cdots, b_0^*), \tag{3.82}$$

and

$$[-\Delta, \Delta] = \{S \in \mathbb{R}^{(2n-1)\times(2n-1)} : -\Delta_{i,j} \leq S_{i,j} \leq \Delta_{i,j}\}, \tag{3.83}$$

where

$$\begin{aligned}
\Delta_{i,j} &= \Delta a_k \text{ if } S(\theta^*)_{i,j} = a_k, k = 0, \cdots, 2n-1,\\
\Delta_{i,j} &= \Delta b_k \text{ if } S(\theta^*)_{i,j} = b_k, k = 0, \cdots, 2n-1\\
\Delta_{i,j} &= 0 \text{ if } S(\theta^*)_{i,j} = 0.
\end{aligned}$$

Now, applying Theorem 3.3.4, we obtain the following result.

Theorem 3.3.5 *Let* Ω *be a subset of* $\mathcal{P}_n$ *and* $\tilde{\Omega}$ *a known symmetric orthotopic set of systems outer-bounding* Ω *defined by* $\{a_i^*, \Delta a_i, b_i^*, \Delta b_i\}_{i=0,\cdots,n-1}$ *according to* (3.80)*. Define the interval Sylvester matrix* $S^* + [-\Delta, \Delta]$ *where the midpoint matrix* S^* *and the* Δ*-matrix are given in* (3.82) *and* (3.83) *respectively. Then*

$$\overline{\sigma}(\Delta) < \underline{\sigma}(S^*) \tag{3.84}$$

implies that $\Omega \subset \mathcal{C}_n$.

Proof The proof directly follows from (3.81), property iii'. and Theorem 3.3.4. ■

3.3.2 A test for strong robustness involving complex structured stability radius

In this subsection, we consider subsets in $\mathcal{C}_n$ and construct a test to check whether this set is strongly robust or not.
The sufficient test for strong robustness of a given set Ω in $\mathcal{C}_n$ given in Theorem 3.2.9 establishes a link between the maximal distance between controllers in $f(\Omega)$ and the minimum of the complex structured stability radii

$$r_{\mathbb{C}}(A(\theta)+B(\theta)f(\theta), B(\theta), I_{2n-1}) \tag{3.85}$$

defined in Definition 3.2.3 over the set Ω. The complex structured stability radius of a Schur matrix under structured perturbations of type (3.85) plays an important role in robustness issues in feedback control analysis [51] and its computation attracted a consequent attention [51], [52], [50], [73], [74]. In particular, we refer to [52] where the following Proposition is proved.

Proposition 3.3.6 *Let $(M, D, E) \in \mathbb{R}^{N\times N} \times \mathbb{R}^{N\times l} \times \mathbb{R}^{q\times N}$. For any $\rho > 0$, define the matrix pencil W_ρ is given by*

$$W_\rho(\lambda) = \begin{bmatrix} M-\lambda I & -\lambda DD^T \\ \rho^2 E^T E & I-\lambda M^T \end{bmatrix},\ \lambda \in \mathbb{C} \tag{3.86}$$

where $\sigma(W_\rho)$ denotes the spectrum of the matrix pencil W_ρ. Then we have

$$r_{\mathbb{C}}(M, D, E) = \min\{\rho \in \mathbb{R}_+ : \sigma(W_\rho) \cap \{s \in \mathbb{C} : |s| = 1\} \neq \emptyset\}. \tag{3.87}$$

From Proposition 3.3.6 we derive the following result.

Theorem 3.3.7 *Let $\Omega \subset \mathcal{C}_n$. For any $\theta \in \Omega$ and any $\rho > 0$ define*

$$W_\rho^\theta(\lambda) = \begin{bmatrix} A(\theta)+B(\theta)f(\theta)-\lambda I & -\lambda B(\theta)(B(\theta))^T \\ \rho^2 I & I-\lambda(A(\theta)+B(\theta)f(\theta))^T \end{bmatrix},\ \lambda \in \mathbb{C}. \tag{3.88}$$

If for all $\theta, \theta' \in \Omega$ we have

$$||f(\theta)-f(\theta')|| < \min\{\rho \in \mathbb{R}_+ : \sigma(W_\rho^\theta) \cap \{s \in \mathbb{C} : |s| = 1\} \neq \emptyset\}, \tag{3.89}$$

where $\sigma(W_\rho^\theta)$ denotes the spectrum of the matrix pencil W_ρ^θ, then Ω is strongly robust.

Proof: Let $\Omega \subset \mathcal{C}_n$. Suppose that (3.89) is satisfied $\forall\theta, \theta' \in \Omega$. Hence using (3.87) with $A = A(\theta)+B(\theta)f(\theta)$, $D = B(\theta)$ and $E = I_{2n-1}$, we obtain that

$$||f(\theta)-f(\theta')|| < r_{\mathbb{C}}(A(\theta)+B(\theta)f(\theta), B(\theta), I_{2n-1}), \forall\theta, \theta' \in \Omega. \tag{3.90}$$

Hence it follows from Theorem 3.2.9 that Ω is strongly robust. ∎

By definition, strong robustness of a set $\Omega \subset \mathcal{C}_n$ requires that the time-varying controller based on any time-varying sequence of systems in Ω yields an asymptotically stable closed-loop system when applied to any system to Ω (see Definition 3.1.2). An interesting achievement of Theorem 3.3.7 is that the time-variation aspect does not appear anymore. Indeed, the condition in Definition 3.1.2 on the set of all possible time-varying closed-loop systems constructed over the set Ω described by

$$x(k+1) = [A(\theta) + B(\theta)f(\theta(k))]x(k), \forall\theta \in \Omega, \forall\{\theta(k)\} \subset \Omega \tag{3.91}$$

has been replaced in (3.89) by a condition on the set of all possible time-invariant closed-loop systems constructed over Ω described by

$$x(k+1) = [A(\theta) + B(\theta)f(\theta')]x(k)\forall\theta, \theta' \in \Omega. \tag{3.92}$$

3.3.3 Strong quadratic robustness and Linear Matrix Inequalities in the case of pole placement

Let us now characterize strongly quadratically robust sets in $\mathcal{C}_n$ by means of Linear Matrix Inequalities (LMI's). We previously saw that strong quadratic robustness is characterized as follows (Definition 3.1.8). A set $\Omega \subset \mathcal{C}_n$ is strongly quadratically robust if there exists a matrix $K = K^T > 0$ in $\mathbb{R}^{(2n-1)\times(2n-1)}$ such that for any system $\theta \in \Omega$ and for any sequence of systems $\{\theta(k)\}_{k\in\mathbb{N}} \subset \Omega$, the following matrix inequality is satisfied:

$$[A(\theta) + B(\theta)f(\theta(k))]^T K[A(\theta) + B(\theta)f(\theta(k))] - K + I < 0. \tag{3.93}$$

However, (3.93) places an infinite number of constraints on Ω. It is our purpose in this section to make additional assumptions on the way systems in $\mathcal{C}_n$ are described and also on the control objective, so as to convert the strong robustness test given in Theorem 3.3.7 into a problem that is numerically tractable. To this effect, we first recall the notion of controller canonical form of a controllable system. make the following assumption.

Definition 3.3.8 (Controller canonical form) *Consider a system $\theta \in \mathcal{C}_n$. Its controller canonical form is defined as follows [93]:*

$$\begin{aligned} x(k+1) &= A^c(\theta)x(k) + B^c(\theta)u(k) \\ y(k) &= C^c(\theta)x(k), \end{aligned} \tag{3.94}$$

where $A^c(\theta) \in \mathbb{R}^{n\times n}$, $B^c(\theta) \in \mathbb{R}^n$ and $C^c(\theta) \in \mathbb{R}^{1\times n}$ are given by:

$$A^c(\theta) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} B^c(\theta) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{3.95}$$

$$C^c(\theta) = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix}, \tag{3.96}$$

where the coefficients a_i, b_i are the coefficients parameterizing the systems in $\mathcal{C}_n$ given by

$$\theta = (a_{n-1}, a_0, \cdots, b_{n-1}, \cdots, b_0)^T. \tag{3.97}$$

Now we introduce the following notation.

Notation 3.3.9 (Set Σ_n of systems in canonical form in $\mathcal{C}_n$) *Let $\Sigma_n \subset \mathcal{C}_n$ be the set of systems in $\mathcal{C}_n$ that assume a controller canonical form* (3.94) *where the state vector $x(k)$ is measurable.*

Notation 3.3.10 (Polyhedral set of systems in canonical form in Σ_n) *Let $S_n \subset \Sigma_n$ be the set of systems in Σ_n whose coefficients a_i and b_i satisfy bounds according to*

$$\underline{a}_i \leq a_i \leq \overline{a}_i \text{ and } \underline{b}_i \leq b_i \leq \overline{b}_i, \ \forall i \leq n-1. \tag{3.98}$$

for known values of the bounds $\underline{a}_i$, $\overline{a}_i$, $\underline{b}_i$ and $\overline{b}_i$. Such a set is called a polyhedral set of systems in Σ_n.

We now make the following assumption.

Assumption 3.3.11 (Pole placement) *The control objective amounts in locating the closed-loop poles in the roots of a given characteristic polynomial $p(\xi) = \xi^n + \sum_{i=0}^{n-1} p_i \xi^i$ with known real coefficients p_i. It is such that for any $\theta \in \Sigma_n$, we have that the controller uniquely defined according to Assumption 3.3.11 can be (uniquely) represented by the feedback law*

$$u(k) = f(\theta)x(k) \tag{3.99}$$

where $f : \Sigma_n \to \mathbb{R}^{1\times n}$ is defined by

$$f(\theta) = \begin{bmatrix} a_0 - p_0 & \dots & a_{n-1} - p_{n-1} \end{bmatrix}. \tag{3.100}$$

Remark 3.3.12 The use of control canonical forms implies that $f(\theta)$ given in (3.100) does not depend on the coefficients b_i. Note that the control objective in Assumption 3.3.11 satisfies Assumption 2.1.13 as we shown in Chapter 2, Section 2.1.3. At first sight, the control law 3.99 does not exactly have the form of the control law given in Assumption 2.1.13. However, choosing the i/s/o description of the system θ given in 2.1.4, we can show that the controller $f(\theta)$ in (3.100) corresponds to a unique controller of the type 2.14, (2.13) satisfying Assumption 2.1.13.

Now, given any pair of systems $\theta, \theta' \in \Sigma_n$, we denote by $(\theta, f(\theta'))$ the closed loop system defined by

$$\begin{aligned} x(k+1) &= A^c(\theta) + B^c(\theta)u(k) \\ u(k) &= f(\theta')x(k) \end{aligned} \tag{3.101}$$

where the closed loop state evolution matrix $A^c(\theta) + B^c(\theta)f(\theta')$ takes the form:

$$A^c(\theta)+B^c(\theta)f(\theta') = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ (a'_0 - a_0) - p_0 & \cdots & & \cdots & (a'_{n-1} - a_{n-1}) - p_{n-1} \end{bmatrix} \tag{3.102}$$

For $i = 0, \ldots, n-1$, let M_i denote the $n \times n$ matrix which is zero except at its (n, i) entry where it is 1. Let $M_n = A^c(\theta^0) + B^c(\theta^0)f(\theta^0)$ denote the nominal closed loop desired matrix. Then, when θ and θ' range over S_n defined in Notation 3.3.10, the matrix $A^c(\theta) + B^c(\theta)f(\theta')$, given in (3.102), is of polyhedral form:

$$M(\delta) := A^c(\theta) + B^c(\theta)f(\theta') = M_n + \sum_{i=0}^{n-1} \delta_i M_i, \tag{3.103}$$

where, for $i = 0, \ldots, n-1$, the parameter δ_i assumes its values in the interval:

$$\underline{\delta}_i := \underline{a}_i - \overline{a}_i \leq \delta_i \leq \overline{a}_i - \underline{a}_i := \overline{\delta}_i \tag{3.104}$$

Let $\delta = \text{col}(\delta_0, \cdots, \delta_{n-1})$ be the uncertainty vector, and define

$$\Delta_0 := \{\delta = \text{col}(\delta_0, \cdots, \delta_{n-1}) \mid \delta_i = \pm\overline{\delta}_i\}, \tag{3.105}$$

$$\Delta := \{\delta = \text{col}(\delta_0, \cdots, \delta_{n-1}) : |\delta_i| \leq \overline{\delta}_i\}. \tag{3.106}$$

Δ_0 is the finite set consisting of all 'corner points' of the uncertainty region (3.104) and Δ is the convex hull of Δ_0. The set of all possible closed-loop state-evolution matrices is defined by the affine set $M(\delta)$ where $\delta \in \Delta$. We have the following result:

Theorem 3.3.13 (Strong quadratic robustness: a finite set of LMI's) *The polyhedral subset $S_n \subset \mathcal{C}_n$ defined in Notation 3.3.10 is strongly quadratically robust if and only if there exists $K = K^T > 0$ such that*

$$[M(\delta)]^T K M(\delta) - K + I < 0, \quad \forall \delta \in \Delta_0 \tag{3.107}$$

where Δ_0 is defined by (3.106).

Proof: to prove the necessity part in Theorem 3.3.13, we go along the following lines. Suppose S_n is as specified. Define Δ_0 according to (3.106). If there exists $K = K^T > 0$ such that (3.107) holds for any $\delta \in \Delta_0$, then convexity of the function $h_x(\delta) := x^T([M(\delta)]^T K M(\delta) - K)x$ for any $x \in \mathbb{R}^n$ implies that [111]

$$[M(\delta)]^T K M(\delta) - K + I < 0, \quad \forall \delta \in \Delta \tag{3.108}$$

Now, define $V : \mathbb{R}^n \to \mathbb{R}$ according to $V(x) = x^T K x$. We then claim that V defines a Lyapunov function for any of the interconnected systems. Indeed, $V(\cdot)$ is non-negative and for any system $\theta, \theta' \in S_n$, the interconnection $(\theta, f(\theta'))$ takes the state-space form $x(k+1) = M(\delta)x(k)$ with $M(\delta)$ defined by (3.103), where $\delta \in \Delta$. Hence we have that

$$\begin{aligned} V(x(k+1)) &= x^T(k) M(\delta)^T K M(\delta) x(k) \\ &< x(k)^T K x(k) = V(x(k)) \end{aligned}$$

for any $\delta \in \Delta$. Therefore, it follows from Definition 3.1.8 that S_n is strongly quadratically robust. Conversely, if no such positive definite matrix K exists, from Definition 3.1.1, there is no quadratic stability of the parameterized closed-loop system and hence S_n is not strongly quadratically robust.This concludes the proof of Theorem 3.3.13. ■

We would like to emphasize the following consequences of Theorem 3.3.13.

1. Theorem 3.3.13 involves a *finite number* of LMI's (at most 2^n LMI's). Therefore, the strong quadratic robustness characterization, placing a priori an infinite number of constraints on the set of systems to be tested, has been converted into a numerically tractable test.

2. In an adaptive control framework, the parameters δ_i range over the diameter of the uncertainty region $|\delta_i| \leq \overline{\delta}_i$, for $i = 0, \cdots, n-1$. Hence, it is an interesting problem to guarantee that the intervals $[\underline{\delta}_i, \overline{\delta}_i]$ will be uniformly decreasing as function of the iteration time of an adaptive algorithm. This amounts to reducing the uncertainty diameters $\overline{a}_i - \underline{a}_i$, $i = 0, \ldots, n-1$. This issue is discussed in Chapter 4.

3. For given uncertainty intervals $\overline{\delta}_i := \overline{a}_i - \underline{a}_i$, the feasibility test of Theorem 3.3.13 depends on the desired pole locations defined by the characteristic polynomial p. This is in accordance with Remark 3.1.15 and shows that some pole locations might be better suited to obtain strong robustness than others.

4. Because of the assumption that the state vector $x(k)$ in (3.94) is measurable, the result presented in Theorem 3.3.13 only holds for a restricted class of polyhedral sets of systems in $\mathcal{C}_n$.

3.3.4 Time-invariant strong robustness and pole placement: a Kharitonov-like test

In this section our purpose is to construct a test to check if a given set $\Omega \subset \mathcal{C}_n$ is time-invariant strongly robust with respect to pole placement in some specified stable poles. It follows from Definition 3.1.10 that a set $\Omega \subset \mathcal{C}_n$ is time invariant strongly robust if for any systems $\theta, \theta' \in \Omega$, the closed-loop characteristic polynomial $\det(\xi I - A(\theta) + B(\theta) f(\theta'))$ is strictly Schur stable, where $A(\theta)$, $B(\theta)$ and $f(\theta')$ are given in (3.18), (3.19) and Assumption 2.1.13 respectively. Hence checking time-invariant strong robustness of Ω comes to the same than checking the Schur stability of the set of all polynomials $\det(\xi I - A(\theta) + B(\theta) f(\theta'))$ when θ, θ' describe Ω. Testing the Hurwitz or Schur stability of a family of polynomials is a relevant question in many stability and robustness problems and led to a large body of literature [22], [60], [57],[86], [85]. In particular, a significant interest has been focused on the issue of Hurwitz stability of a polynomial interval which first appeared in [57], leading to the celebrated Kharitonov's theorem which we will recall further. Counterparts of this result for testing Schur stability of a polynomial interval can be found in [86], [85], [60], [22] but still suffer from a much higher computational complexity.
Using these results, our objective is now to express a sufficient test for time-invariant strong robustness of bounded orthotopic sets of systems in $\mathcal{C}_n$ in the form of a a Kharitonov-like test [26].
We first recall the Kharitonov's Theorem to test the Hurwitz stability of interval polynomials [57].

Theorem 3.3.14 (The Kharitonov's Theorem in the continuous-time description)
For all $N \in \mathbb{N}$, each member of the infinite family of polynomials

$$\chi(\xi) = \chi_0 + \chi_1 \xi + \cdots + \chi_N \xi^N \tag{3.109}$$

with

$$\underline{\chi_i} \leq \chi_i \leq \overline{\chi_i},\ \forall i = 0, \cdots, N, \tag{3.110}$$

where $(\underline{\chi_i}, \overline{\chi_i})_{i=0,\cdots,N}$ *are given constants, is strictly Hurwitz if and only if each of the four Kharitonov polynomials*

$$\begin{aligned}
\gamma_1(\xi) =& \underline{\chi_0} + \underline{\chi_1}\xi + \overline{\chi_2}\xi^2 + \overline{\chi_3}\xi^3 + \underline{\chi_4}\xi^4 + \underline{\chi_5}\xi^5 + \overline{\chi_6}\xi^6 + \cdots \\
\gamma_2(\xi) =& \overline{\chi_0} + \overline{\chi_1}\xi + \underline{\chi_2}\xi^2 + \underline{\chi_3}\xi^3 + \overline{\chi_4}\xi^4 + \overline{\chi_5}\xi^5 + \underline{\chi_6}\xi^6 + \cdots \\
\gamma_3(\xi) =& \overline{\chi_0} + \underline{\chi_1}\xi + \underline{\chi_2}\xi^2 + \overline{\chi_3}\xi^3 + \overline{\chi_4}\xi^4 + \underline{\chi_5}\xi^5 + \underline{\chi_6}\xi^6 + \cdots \\
\gamma_4(\xi) =& \underline{\chi_0} + \overline{\chi_1}\xi + \overline{\chi_2}\xi^2 + \underline{\chi_3}\xi^3 + \underline{\chi_4}\xi^4 + \overline{\chi_5}\xi^5 + \overline{\chi_6}\xi^6 + \cdots
\end{aligned}$$

is strictly Hurwitz.

We consider orthotopic sets of systems, also called boxes of systems. These sets are defined as follows.

Definition 3.3.15 (Boxes of systems in $\mathcal{C}_n$) *We call box of systems in $\mathcal{C}_n$ any set $\mathcal{I}_n \subset \mathcal{C}_n$ associated to the $4n$ given constants $\{\underline{a_i}, \overline{a_i}, \underline{b_i}, \overline{b_i}\}_{i=0,\cdots,n-1}$ such that for all system $\theta \in \mathcal{I}_n$, the parameters a_i, b_i given in* (4.6) *satisfy:*

$$a_i \in [\underline{a_i}, \overline{a_i}] \text{ and } b_i \in [\underline{b_i}, \overline{b_i}],\ \forall i = 0, \cdots, n-1. \tag{3.111}$$

We consider pole placement in some pre-specified stable poles $\{\alpha_i\}_{i\in\mathbb{N}}$, $|\alpha_i| < 1$, $\forall i$. We will use the following notation.

Notation 3.3.16 (Controllers) *Any system in $\theta \in \mathcal{C}_n$ is described by the input/output equation in discrete-time*

$$y(k+1) + \sum_{i=0}^{n-1} a_i y(k+i-n+1) = \sum_{i=0}^{n-1} b_i u(k+i-n+1) \tag{3.112}$$

and is associated to the polynomials given by $\mathcal{A}(\xi) = \xi^n + a_{n-1}\xi^{n-1} + \cdots + a_0$ and $\mathcal{B}(\xi) = b_{n-1}\xi^{n-1} + \cdots + b_0$.

For any system $\theta \in \mathcal{C}_n$, its unique controller $f(\theta)$ is identified with its parameter vector

$$f(\theta) := (c_0, \cdots, c_{n-2}, d_0, \cdots, d_{n-1})^T \in \mathbb{R}^{2n-1}. \tag{3.113}$$

This controller is described by

$$u(k) + \sum_{i=0}^{n-2} c_i u(k+i-n+1) = \sum_{i=0}^{n-2} d_i y(k+i-n+1). \tag{3.114}$$

With the controller (3.114) we associate the polynomials $\mathcal{C}(\xi) = \xi^{n-1} + c_{n-2}\xi^{n-2} + \cdots + c_0$ and $\mathcal{D}(\xi) = d_{n-2}\xi^{n-2} + \cdots + d_0$.

We now introduce the following notation.

Notation 3.3.17 (Characteristic polynomial) *For any systems $\theta, \theta' \in \mathcal{C}_n$, we denote by $\chi_{\theta,f(\theta')(\xi)}$ the characteristic polynomial of the closed-loop system defined by*

$$x(k+1) = A(\theta)x(k) + B(\theta)u(k) \tag{3.115}$$

$$u(k) = f(\theta')x(k), \tag{3.116}$$

where $A(\theta)$, $B(\theta)$, $f(\theta')$ and $x(k)$ are defined in (3.18), (3.19), (3.6) *and Assumption 2.1.13 respectively. Hence, we have*

$$\chi_{\theta,f(\theta')(\xi)} = \det(\xi I - (A(\theta) + B(\theta)f(\theta'))) \tag{3.117}$$

$$= \mathcal{A}(\xi)\mathcal{C}(\xi) + \mathcal{B}(\xi)\mathcal{D}(\xi). \tag{3.118}$$

where $\mathcal{A}(\xi)$, $\mathcal{B}(\xi)$, $\mathcal{C}(\xi)$ and $\mathcal{D}(\xi)$, are defined in Notation 3.3.16. Hence $\forall\theta, \theta^0 \in \mathcal{C}_n$, we have

$$\chi_{\theta,\phi(\theta^0)}(\xi) = (\xi^n + \sum_{i=0}^{n-1} a_i\xi^i)(\xi^{n-1} + \sum_{i=0}^{n-2} c_i^0\xi^i) + (\sum_{i=0}^{n-1} b_i\xi^i)(\sum_{i=0}^{n-1} d_i^0\xi^i). \tag{3.119}$$

$$\textit{Denoting } \chi_{\theta,\phi(\theta^0)}(\xi) = \xi^{2n-1} + \sum_{i=0}^{2n-2} \chi_i\xi^i, \tag{3.120}$$

we associate the polynomial $\chi_{\theta,\phi(\theta^0)}(\xi)$ with its coefficient vector

$$\underline{\chi}_{\theta,\phi(\theta^0)} = (1, \chi_{2n-2}, \cdots, \chi_0)^T \in \mathbb{R}^{2n}. \tag{3.121}$$

The construction of a test to check whether a box of systems in $\mathcal{C}_n$ as defined in Definition 3.3.15 is time-invariant strongly robust or not is illustrated in Figure 3.5. It goes along the following steps.

Algorithm 3.3.18 (Time invariant strong robustness: a sufficient Kharitonov-like test)

(1) **Set of characteristic polynomials generated by a fixed controller:** fix a system $\theta^0 \in \mathcal{I}_n$ and characterize the set of characteristic polynomials associated with the set of closed-loop systems $\{(\theta, f(\theta^0))\}_{\theta\in\mathcal{I}_n}$:

$$\Gamma_{\theta^0}(\mathcal{I}_n) = \{\chi_{\theta,f(\theta^0)}(\xi), \theta \in \mathcal{I}_n\} \subset \mathbb{R}^{2n}[\xi] \tag{3.122}$$

(2) **Relation between Schur stability and Hurwitz stability:** transform the set $\Gamma_{\theta^0}(\mathcal{I}_n)$ into a set $\tilde{\Gamma}_{\theta^0}(\mathcal{I}_n) \subset \mathbb{R}^{2n}[\xi]$ defined by

$$\tilde{\Gamma}_{\theta^0}(\mathcal{I}_n) = \{\tilde{\chi}(\xi) := (\xi-1)^n\chi(\frac{\xi+1}{\xi-1}) : \chi(\xi) \in \Gamma_{\theta^0}(\mathcal{I}_n)\} \tag{3.123}$$

In [13], it is proven that for any $\chi(\xi) \in \Gamma_{\theta^0}(\mathcal{I}_n)$, $\chi(\xi)$ has all its zeros within the open unit complex disc if and only if the associated polynomial $\tilde{\chi}(\xi)$ defined in (3.123) has all its zeros in the open half complex plane.

(3) **Closure box of $\mathcal{I}_n$ associated with a fixed controller:** because the transformed set $\tilde{\Gamma}_{\theta^0}(\mathcal{I}_n)$ is in general not a box, define an outer bounding box of systems $\tilde{\Gamma}^+_{\theta^0}(\mathcal{I}_n)$ of $\tilde{\Gamma}_{\theta^0}(\mathcal{I}_n)$, called *closure box of* $\tilde{\Gamma}_{\theta^0}(\mathcal{I}_n)$.

(4) **Maximal closure box:** generate the set $\bigcup_{\theta^0 \in \mathcal{I}_n} \tilde{\Gamma}_{\theta^0}(\mathcal{I}_n)$ and compute an outer bounding box $\tilde{\Gamma}^+(\mathcal{I}_n)$ for this set. The box $\Gamma^+(\mathcal{I}_n)$ is called the maximal closure box of $\mathcal{I}_n$.

(5) **Kharitonov's test:** apply the Kharitonov's test given in Theorem 3.3.14 on the polynomial interval $\tilde{\Gamma}^+(\mathcal{I}_n)$. If the test is positive then any characteristic polynomial in $\tilde{\Gamma}^+(\mathcal{I}_n)$ is strictly Hurwitz stable, therefore any characteristic polynomial in $\Gamma^+(\mathcal{I}_n)$ is strictly Schur stable. Equivalently, the closed loop system $(\theta, f(\theta^0))$ is asymptotically stable for any $\theta, \theta^0 \in \mathcal{I}_n$, consequently $\mathcal{I}_n$ is time invariant strongly robust.

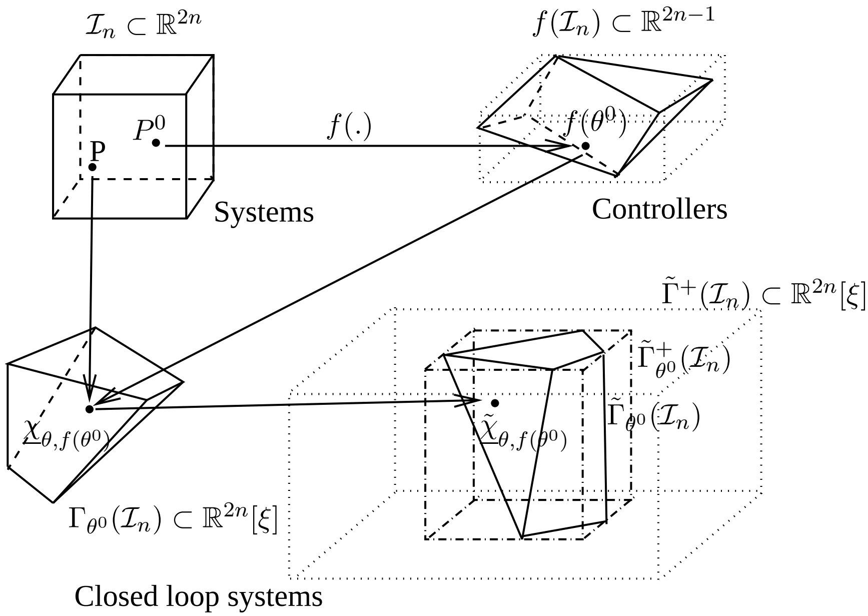


Figure 3.5: Strong robustness and boxes of systems, discrete time description.

These various steps are now discussed in more details. In the next discussions, $\mathcal{I}_n$ denotes a given box of systems in $\mathcal{C}_n$ as defined in Definition 3.3.15.

Set of characteristic polynomials generated by a fixed controller

We suppose the system $\theta^0 = (a^0_{n-1}, \cdots, a^0_0, b^0_{n-1}, \cdots, b^0_0)^T \in \mathcal{I}_n$ to be fixed. The known desired closed loop characteristic polynomial is denoted by

$$\chi^0(\xi) = \prod_{i=1}^{2n-1} (\xi - \alpha_i), \tag{3.124}$$

and is associated with its coefficient vector $\underline{\chi}^0$. The controller based on θ^0 is $f(\theta^0) = (c^0_{n-2}, \cdots, c^0_0, d_{n-2},^0 \cdots, d^0_0)^T$ defined as the unique solution of

$$\chi_{\theta^0, f(\theta^0)}(\xi) = \chi^0(\xi). \tag{3.125}$$

according (3.119). The set $\Gamma_{\theta^0}(\mathcal{I}_n)$ of all characteristic polynomials generated by $f(\theta^0)$ is the set of polynomials $\{\chi_{\theta,f(\theta^0)}(\xi)\}_{\theta\in\mathcal{I}_n}$ where $\chi_{\theta,f(\theta^0)}(\xi)$ is defined by (3.119).

Relation between Schur stability and Hurwitz stability

Our ultimate goal is to use the Kharitonov's Theorem for continuous systems to check the stability of any characteristic polynomial $\chi_{\theta,f(\theta^0)}(\xi), \theta, \theta^0 \in \mathcal{I}_n$. Since we deal with systems in discrete-time description, we first transform the problem of testing the Schur stability of a polynomial into the problem of testing the Hurwitz stability of a related polynomial. We first recall that any real polynomial $p(\xi) = p_0 + \cdots + p_N\xi^N$ has all its zeros within the open unit disc if and only if the polynomial $\tilde{p}(\xi) = (\xi-1)^N p(\frac{\xi+1}{\xi-1})$ has all its zeros in the open-half plane [13]. Therefore, the Schur stability of the polynomial described by (3.119) is equivalent to the Hurwitz stability of

$$\begin{aligned}\tilde{\chi}_{\theta,f(\theta^0)}(\xi) =&(\sum_{i=0}^{n} a_i(\xi+1)^i(\xi-1)^{n-i})(\sum_{j=0}^{n-1} c_j^0(\xi+1)^j(\xi-1)^{n-1-j})\\ &+(\xi-1)(\sum_{k=0}^{n-1} b_k^0(\xi+1)^k(\xi-1)^{n-1-k})(\sum_{l=0}^{n-1} d_l^0(\xi+1)^l(\xi-1)^{n-1-l})\end{aligned} \tag{3.126}$$

where $a_n = 1$ and $c_{n-1}^0 = 0$. This polynomial can be re-written as:

$$\tilde{\chi}_{\theta,f(\theta^0)}(\xi) = \sum_{i=0}^{2n-1} \tilde{\chi}_i\xi^i = (\sum_{i=0}^{2n-1} \tilde{a_i}\xi^i)(\sum_{j=0}^{n-1} \tilde{c_j^0}\xi^j) + (\xi-1)(\sum_{k=0}^{n-1} \tilde{b_k}\xi^k)(\sum_{l=0}^{n-1} \tilde{d_l^0}\xi^l), \tag{3.127}$$

where the coefficients $\tilde{a_i}, \tilde{b_i}, \tilde{c_i}, \tilde{d_i}$ are calculated according to [13] as follows. For the coefficients $\tilde{a_i}$ we get:

$$[\tilde{a_n}, \cdots, \tilde{a_0}] = [a_n, a_{n-1}, \cdots, a_0]\Gamma_{n+1} \tag{3.128}$$

where the $(n+1)\times(n+1)$ matrix $\Gamma_{n+1} = [\gamma_{i,j}]_{i,j=1,\cdots,n+1}$ is given by the formula

$$\gamma_{i,j} = \gamma_{i,j+1} + \gamma_{i-1,j+1} + \gamma_{i-1,j}, i = 2,\cdots,n \text{ and } j = n,\cdots,1,$$

$$\text{subject to } \gamma_{i,n} = 1, i = 1,\cdots,n+1,$$

and where the element $\gamma_{1,j}$ is the binomial coefficient in μ^j in the expansion of $(\mu-1)^n$ for $j = 1,\cdots,n+1$. The same result applies to the computation of $\tilde{b_i}, \tilde{c_i}$ and $\tilde{d_i}$. Denoting the polynomial $\tilde{\chi}_{\theta,f(\theta^0)}(\xi)$ in $\tilde{\Gamma}^+(\mathcal{I}_n)$ by

$$\tilde{\chi}_{\theta,f(\theta^0)}(\xi) = \sum_{i=0}^{2n-1} \tilde{\chi}_i\xi^i, \tag{3.129}$$

we associate with $\tilde{\chi}_{\theta,f(\theta^0)}(\xi)$ its coefficient vector

$$\underline{\tilde{\chi}}_{\theta,f(\theta^0)} = (\underline{\tilde{\chi}}_{\theta,f(\theta^0),2n-2},\cdots,\underline{\tilde{\chi}}_{\theta,f(\theta^0),2n-1})^T \in \mathbb{R}^{2n}. \tag{3.130}$$

To conclude, for some given $\theta, \theta^0 \in \mathcal{I}_n$, the problem of checking the Schur stability of the polynomial $\chi_{\theta,f(\theta^0)}(\xi)$ has been transformed into checking the Hurwitz stability of the related polynomial $\tilde{\chi}_{\theta,f(\theta^0)}(\xi)$ defined by (3.127).

Closure box of $\mathcal{I}_n$ associated with a fixed system: definition

The closure box $\tilde{\Gamma}^+_{\theta^0}(\mathcal{I}_n)$ of $\mathcal{I}_n$ associated with θ^0 is the outer bounding box of $\Gamma_{\theta^0}(\mathcal{I}_n)$ defined by:

$$\Gamma^+_{\theta^0}(\mathcal{I}_n) = (x_{2n-2}, x_{2n-3}, \cdots, x_0)^T \in \mathbb{R}^{2n-1} : x_i \in [\underline{X_i}(\theta^0), \overline{X_i}(\theta^0)]\} \tag{3.131}$$

$$\text{where } \underline{X_i}(\theta^0) = \min_{\theta\in\mathcal{I}_n} \tilde{\chi}_{\theta,f(\theta^0),i} \text{ and } \overline{X_i}(\theta^0) = \max_{\theta\in\mathcal{I}_n} \tilde{\chi}_{\theta,f(\theta^0),i}. \tag{3.132}$$

The set of boxes $\{\tilde{\Gamma}^+_{\theta^0}(\mathcal{I}_n)\}_{\theta^0\in\mathcal{I}_n}$ is hence a set of parallel boxes.

Maximal closure box of $\mathcal{I}_n$

The Maximal Closure Box $\tilde{\Gamma}^+(\mathcal{I}_n)$ of $\mathcal{I}_n$ is defined as the minimal box $\tilde{\Gamma}^+(\mathcal{I}_n)$ enclosing the closure boxes $\tilde{\Gamma}^+_{\theta^0}(\mathcal{I}_n)$, $\forall\theta^0 \in \mathcal{I}_n$:

$$\tilde{\Gamma}^+(\mathcal{I}_n) = \{(x_{2n-2}, x_{2n-3}, \cdots, x_0)^T \in \mathbb{R}^{2n-1} : x_i \in [\underline{X_i}, \overline{X_i}]\} \tag{3.133}$$

where

$$\underline{X_i} = \min_{\theta^0\in\mathcal{I}_n} \underline{X_i}(\theta^0) \text{ and } \overline{X_i} = \max_{\theta^0\in\mathcal{I}_n} \overline{X_i}(\theta^0). \tag{3.134}$$

Combining (3.132) and (3.134) leads to

$$\underline{X_i} = \min_{\theta^0\in\mathcal{I}_n} \{\min_{P\in\mathcal{I}_n} \tilde{\chi}_{\theta,f(\theta^0),i}\} \text{ and } \overline{X_i} = \max_{\theta^0\in\mathcal{I}_n} \{\max_{\theta\in\mathcal{I}_n} \tilde{\chi}_{\theta,f(\theta^0),i}\}. \tag{3.135}$$

Equivalently,

$$\underline{X_i} = \min_{\theta\in\mathcal{I}_n}\{\min_{Y\in f(\mathcal{I}_n)} \tilde{\chi}_{\theta,Y,i}\} \text{ and } \overline{X_i} = \max_{\theta\in\mathcal{I}_n}\{\max_{Y\in f(\mathcal{I}_n)} \tilde{\chi}_{\theta,Y,i}\}. \tag{3.136}$$

Our aim is to compute the dimensions $\underline{X_i}$ and $\overline{X_i}$ of $\tilde{\Gamma}^+(\mathcal{I}_n)$. To this end, we first compute a minimal box enclosing $f(\mathcal{I}_n)$. Then, we compute the values for $\underline{X_i}$ and $\overline{X_i}$ using (3.136).

1. Computation of a box enclosing $f(\mathcal{I}_n)$: the problem consists in finding some conservative bounds on the coefficients of the controller $f(\theta)$ when θ describes $\mathcal{I}_n$. Equivalently, our aim is to find the maximal variations induced on the coefficients c_i, d_i of the polynomials $\mathcal{C}(\xi)$ and $\mathcal{D}(\xi)$ such that the equality

$$\mathcal{A}(\xi)\mathcal{C}(\xi) + \mathcal{B}(\xi)\mathcal{D}(\xi) = \chi^0(\xi) \tag{3.137}$$

holds, where the coefficients a_i, b_i of $\mathcal{A}(\xi)$ and $\mathcal{B}(\xi)$ belong to $\mathcal{I}_i = [\underline{a_i}, \overline{a_i}]$ and $\mathcal{J}_i = [\underline{b_i}, \overline{b_i}]$ respectively for any $i = 0, \cdots, n-1$. (3.137) can be written as:

$$M.X = \underline{\chi}^0 \tag{3.138}$$

$$\text{with } M = \begin{pmatrix} A & B \end{pmatrix} \text{ and } X = \begin{pmatrix} C \\ D \end{pmatrix} \tag{3.139}$$

where $A, B \in \mathbb{R}^{2n\times n}$ and $C, D \in \mathbb{R}^n$ are given by:

$$A = \begin{bmatrix} 1 & 0 & 0 & & 0 \\ a_{n-1} & 1 & 0 & & \\ a_{n-2} & a_{n-1} & 1 & & 0 \\ \vdots & a_{n-2} & a_{n-1} & & 1 \\ a_0 & \vdots & a_{n-2} & & a_{n-1} \\ 0 & a_0 & \vdots & & a_{n-2} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & \cdots & a_0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 & & 0 \\ b_{n-1} & 1 & 0 & & \\ b_{n-2} & b_{n-1} & 1 & & 0 \\ \vdots & b_{n-2} & b_{n-1} & & 1 \\ b_0 & \vdots & b_{n-2} & & b_{n-1} \\ 0 & b_0 & \vdots & & b_{n-2} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & \cdots & b_0 \end{bmatrix}$$

$$C^T = \begin{pmatrix} 1 & c_{n-2} & c_{n-3} & \cdots & c_0 \end{pmatrix}, \; D^T = \begin{pmatrix} d_{n-1} & d_{n-2} & \cdots & d_0 \end{pmatrix}$$

and where $\underline{\chi}^0 = (1, \chi^0_{2n-2}, \cdots, \chi^0_0)^T \in \mathbb{R}^{2n}$ is defined by (3.124). The problem is now the following: if the coefficients of M vary within $\mathcal{I}_i$, $\mathcal{J}_i$ for $i = 0 \cdots n-1$, what is the set described by the solutions X of (3.138)?

Let M^0 denote the matrix of the form (3.139), associated with a nominal system $\theta^0 \in \mathcal{I}_n$. Let $M^0 + \delta M$ denotes the perturbed matrix corresponding to M^0, where δM can be all admissible perturbation matrix so that the coefficients $a_i^0 + \delta a_i$, $b_i^0 + \delta b_i$ stay in the segments $\mathcal{I}_i$, $\mathcal{J}_i$ respectively for $i = 0, \cdots, n-1$. We denote by $\mathcal{M}$ the corresponding perturbation set described by δM. We call $X^0 + \delta X$ the controller associated to the perturbed system leading to the matrix $M^0 + \delta M$. We have then:

$$M^0.X^0 = \underline{\chi}^0 \tag{3.140}$$

$$\text{and } (M^0 + \delta M).(X^0 + \delta X) = \underline{\chi}^0. \tag{3.141}$$

Subtracting (3.140) from (3.141) yields:

$$(M^0 + \delta M).\delta X = -\delta M.X^0 \tag{3.142}$$

Now, since $\mathcal{I}_n$ contains only controllable systems, we know that $M^0 + \delta M$ is non-singular for all $\delta M \in \mathcal{D}$. Hence (3.142) can be written as follows:

$$\delta X = -(M^0 + \delta M)^{-1}.\delta M.X^0 \tag{3.143}$$

Hence:

$$\|\delta X\|_2 = \|(M^0 + \delta M)^{-1}.\delta M.X^0\|_2 \tag{3.144}$$

Therefore:

$$\|\delta X\|_2 \leq \|(M^0 + \delta M)^{-1}\|_{I2}.\|\delta M\|_{I2}.\|X^0\|_2 \tag{3.145}$$

where $\|.\|_{I2}$ denotes the 2-norm induced norm in $\mathbb{R}^{2n\times 2n}$. Consequently:

$$\|\delta X\|_2 \leq \overline{\sigma}\{(M^0 + \delta M)^{-1}\}.\overline{\sigma}\{\delta M\}.\|X^0\|_2 \tag{3.146}$$

where $\overline{\sigma}(T)$ denotes the largest eigenvalue of the matrix $T^T T$. Then, since

$$\overline{\sigma}\{(M^0 + \delta M)^{-1}\} = \frac{1}{\underline{\sigma}\{(M^0 + \delta M)\}}, \tag{3.147}$$

with $\underline{\sigma}(T)$ denoting the smallest eigenvalue of $T^T T$, (3.146) yields:

$$\|\delta X\|_2 \leq \frac{\overline{\sigma}\{\delta M\}.\|X^0\|_2}{\underline{\sigma}\{(M^0+\delta M)\}} \tag{3.148}$$

Now, using the two properties

$$\underline{\sigma}\{(M^0+\delta M)\} \geq \underline{\sigma}(M^0) - \overline{\sigma}(\delta M) \tag{3.149}$$

and

$$\overline{\sigma}\{\delta M\} \leq \sqrt{2}\max[\overline{\sigma}(\delta A), \overline{\sigma}(\delta B)], \tag{3.150}$$

(3.148) leads to:

$$\|\delta X\|_2 \leq \frac{\sqrt{2}\|X^0\|_2 \max\{\overline{\sigma}(\delta A), \overline{\sigma}(\delta B)\}}{\underline{\sigma}(M^0) - \sqrt{2}\max\{\overline{\sigma}(\delta A), \overline{\sigma}(\delta B)\}} \tag{3.151}$$

Therefore, any controller vector $X^0+\delta X$ solution of (3.141) satisfies

$$\|\delta X\|_2 \leq \max_{\delta M \in \mathcal{D}} \frac{\sqrt{2}\|X^0\|_2 \max\{\overline{\sigma}(\delta A), \overline{\sigma}(\delta B)\}}{\underline{\sigma}(M^0) - \sqrt{2}\max\{\overline{\sigma}(\delta A), \overline{\sigma}(\delta B)\}} \tag{3.152}$$

which gives bounds on the coefficients c_i, d_i of the controller associated with any perturbed systems in $\mathcal{I}_n$. Since

$$\|\delta X\|_2^2 = \sum_{i=0}^{n-2} \delta c_i^2 + \sum_{j=0}^{n-1} \delta d_j^2, \tag{3.153}$$

we obtain, for any $i = 0, \cdots, n-2$:

$$|\delta c_i| \leq \frac{\sqrt{2}\|X^0\|_2 \max_{\delta M \in \mathcal{D}}\{\max\{\overline{\sigma}(\delta A), \overline{\sigma}(\delta B)\}\}}{\underline{\sigma}(M^0) - \sqrt{2}\max_{\delta M \in \mathcal{D}}\{\max\{\overline{\sigma}(\delta A), \overline{\sigma}(\delta B)\}\}}, \tag{3.154}$$

and for any $j = 0, \cdots, n-1$:

$$|\delta d_j| \leq \frac{\sqrt{2}\|X^0\|_2 \max_{\delta M \in \mathcal{D}}\{\max\{\overline{\sigma}(\delta A), \overline{\sigma}(\delta B)\}\}}{\underline{\sigma}(M^0) - \sqrt{2}\max_{\delta M \in \mathcal{D}}\{\max\{\overline{\sigma}(\delta A), \overline{\sigma}(\delta B)\}\}}. \tag{3.155}$$

Summary: fix $\theta^0 = (a_0^0, \cdots, a_{n-1}^0, b_0^0, \cdots, b_{n-1}^0)^T$ a system in $\mathcal{I}_n$. Denote by $\delta\theta = (\delta a_0, \cdots, \delta a_{n-1}, \delta b_0, \cdots, \delta b_{n-1})^T$ any perturbation affecting θ^0 leaving the system $\theta^0+\delta\theta$ in $\mathcal{I}_n$. Call $M^0 = \begin{pmatrix} A^0 & B^0 \end{pmatrix}$ the matrix associated with θ^0 following (3.139). The perturbed matrix M is then $M^0+\delta M = \begin{pmatrix} A+\delta A & B+\delta B \end{pmatrix}$. Let X^0 be the vector associated to the controller $f(\theta^0) = (c_0^0, \cdots, c_{n-2}^0, d_0^0, \cdots, d_{n-1}^0)^T$ and defined in (3.139). Finally let δX be the perturbation induced on X^0 when θ^0 is perturbed of $\delta\theta$. Then the coefficients of the perturbation vector δX have the following bounds:

$$|\delta c_i| \leq \frac{\sqrt{2}\|X^0\|_2 \max_{\delta M \in \mathcal{M}}\{\max\{\overline{\sigma}(\delta A), \overline{\sigma}(\delta B)\}\}}{\underline{\sigma}(M^0) - \sqrt{2}\max_{\delta M \in \mathcal{M}}\{\max\{\overline{\sigma}(\delta A), \overline{\sigma}(\delta B)\}\}}, i \leq n-2,$$

$$|\delta d_j| \le \frac{\sqrt{2}\|X^0\|_2 \max_{\delta M\in\mathcal{M}}\{\max\{\overline{\sigma}(\delta A),\overline{\sigma}(\delta B)\}\}}{\underline{\sigma}(M^0)-\sqrt{2}\max_{\delta M\in\mathcal{M}}\{\max\{\overline{\sigma}(\delta A),\overline{\sigma}(\delta B)\}\}}, j \le n-1.$$

Finally, for any system $\theta \in \mathcal{I}_n$, the controller $f(\theta) = (c_0, \cdots, c_{n-2}, d_0, \cdots, d_{n-1})^T$ associated with θ is such that

$$|c_i - c_i^0| \le \frac{\sqrt{2}\|X^0\|_2 \max_{\delta M\in\mathcal{M}}\{\max\{\overline{\sigma}(\delta A),\overline{\sigma}(\delta B)\}\}}{\underline{\sigma}(M^0)-\sqrt{2}\max_{\delta M\in\mathcal{M}}\{\max\{\overline{\sigma}(\delta A),\overline{\sigma}(\delta B)\}\}}, i \le n-2 \quad (3.156)$$

and

$$|d_j - d_j^0| \le \frac{\sqrt{2}\|X^0\|_2 \max_{\delta M\in\mathcal{M}}\{\max\{\overline{\sigma}(\delta A),\overline{\sigma}(\delta B)\}\}}{\underline{\sigma}(M^0)-\sqrt{2}\max_{\delta M\in\mathcal{M}}\{\max\{\overline{\sigma}(\delta A),\overline{\sigma}(\delta B)\}\}}, j \le n-1 \quad (3.157)$$

where the coefficients c_i^0, d_j^0 are computed according to subsection 3.3.4.

2. Computation of $\underline{X_i}$ and $\overline{X_i}$: this step consists in finding some bounds on the coefficients of the polynomial $\tilde{\chi}_{\theta,f(\theta^0)}(\xi)$ when both θ and θ^0 describe $\mathcal{I}_n$. This is equivalent in finding some bounds on the coefficients of the vector $\underline{\tilde{\chi}}$ solution of

$$\begin{pmatrix}\tilde{A} & \tilde{B}\end{pmatrix}\begin{pmatrix}\tilde{C^0} \\ \tilde{D^0}\end{pmatrix} = \underline{\tilde{\chi}}, \quad (3.158)$$

where $\tilde{A}, \tilde{B}, \tilde{C^0}$ and $\tilde{D^0}$ have the same structure as A, B, C and D previously defined, and their coefficients $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i^0, \tilde{d}_i^0$ are calculated using (3.128). We denote by $\underline{\tilde{\chi}}$ the coefficient vector associated to the transformed closed-loop polynomial:

$$\tilde{\chi}(\xi) = (\xi-1)^{2n-1}\chi(\frac{\xi+1}{\xi-1}) \quad (3.159)$$

introduced in subsection 3.3.4.
(3.158) is a system of $2n$ equations with the $4n-1$ coefficients $\{\tilde{a}_i\}_{i=0,\cdots,n-1}$, $\{\tilde{b}_i\}_{i=0,\cdots,n-1}$, $\{\tilde{d}_i^0\}_{i=0,\cdots,n-1}$ and $\{\tilde{c}_i^0\}_{i=0,\cdots,n-2}$. These $4n-1$ parameters lie in the box $\tilde{\mathcal{P}}$ defined by

$$\tilde{\mathcal{P}} = \{(\tilde{a}_0, \cdots, \tilde{a}_{n-1}, b_0, \cdots, \tilde{b}_{n-1}, \tilde{c}_0, \cdots, \tilde{c}_{n-2}, \tilde{d}_0, \cdots, \tilde{d}_{n-1}) \in \mathbb{R}^{4n-1}\}$$

where $\tilde{a}_i \in [\underline{\tilde{a}_i}, \overline{\tilde{a}_i}]$, $\tilde{b}_i \in [\underline{\tilde{b}_i}, \overline{\tilde{b}_i}]$, $\tilde{c}_i^0 \in [\underline{\tilde{c}_i^0}, \overline{\tilde{c}_i^0}]$ and $\tilde{d}_i^0 \in [\underline{\tilde{d}_i^0}, \overline{\tilde{d}_i^0}]$. The bounds $\underline{\tilde{a}_i}$, $\overline{\tilde{a}_i}$, $\underline{\tilde{b}_i}$, $\overline{\tilde{b}_i}$ are constant values determined by the dimensions $\underline{a_i}, \overline{a_i}, \underline{b_i}, \overline{b_i}$ of the box $\mathcal{I}_n$ according (3.128). The bounds $\underline{\tilde{c}_i^0}$, $\overline{\tilde{c}_i^0}$, $\underline{\tilde{d}_i^0}$, $\overline{\tilde{d}_i^0}$ are computed following (3.128), (3.156) and (3.157).
Each of the $2n$ equations of (3.158) is linear in its $4n-1$ coefficients and is defined on the compact set $\tilde{\mathcal{P}}$, hence the minimum and the maximum of the coefficients $\underline{\tilde{\chi}}_i$ are reached on one of the corner of $\tilde{\mathcal{P}}$ for any $i = 0, \cdots, 2n-1$.
Therefore, the computation of the i^{th} equation in (3.158) on each of the $4n-1$ corners of the box $\tilde{\mathcal{P}}$ leads to the family $\{\tilde{\chi}_i^j\}_{j=1,\cdots,4n-1}$ of $4n-1$ candidate values for the

coefficient $\underline{\tilde{\chi}}_i$ of $\underline{\tilde{\chi}}$. We finally compute the maximum and the minimum value of this collection to obtain a lower and upper bound on $\underline{\tilde{\chi}}_i$:

$$\underline{X_i} = \min_{j=1,\cdots,4n-1} \underline{\tilde{\chi}}_i^j \text{ and } \overline{X_i} = \max_{j=1,\cdots,4n-1} \underline{\tilde{\chi}}_i^j,\ i = 1, \cdots, 2n-1, \tag{3.160}$$

which define the dimensions of $\tilde{\Gamma}^+(\mathcal{I}_n)$. The computation of the maximal closure box $\tilde{\Gamma}^+(\mathcal{I}_n)$ of $\mathcal{I}_n$ is therefore completed. The next and last step is to apply the Kharitonov's theorem on $\tilde{\Gamma}^+(\mathcal{I}_n)$ to conclude about the strong robustness of $\mathcal{I}_n$.

Strong Kharitonov's test

First remark that for any θ, θ^0 in $\mathcal{I}_n$, the closed-loop polyonimal $\tilde{\chi}_{\theta,f(\theta^0)}(\xi)$ is contained in the maximal closure box $\tilde{\Gamma}^+(\mathcal{I}_n)$. As a consequence, if any element in $\tilde{\Gamma}^+(\mathcal{I}_n)$ is strictly Hurwitz stable, then $\mathcal{I}_n$ is strongly robust. To complete this step we use Theorem 3.3.14. We define the four polynomials:

$$\begin{aligned}
\gamma_1(\xi) =& \underline{X_0} + \underline{X_1}\xi + \overline{X_2}\xi^2 + \overline{X_3}\xi^3 + \underline{X_4}\xi^4 + \underline{X_5}\xi^5 + \overline{X_6}\xi^6 + \cdots \\
\gamma_2(\xi) =& \overline{X_0} + \overline{X_1}\xi + \underline{X_2}\xi^2 + \underline{X_3}\xi^3 + \overline{X_4}\xi^4 + \overline{X_5}\xi^5 + \underline{X_6}\xi^6 + \cdots \\
\gamma_3(\xi) =& \overline{X_0} + \underline{X_1}\xi + \underline{X_2}\xi^2 + \overline{X_3}\xi^3 + \overline{X_4}\xi^4 + \underline{X_5}\xi^5 + \underline{X_6}\xi^6 + \cdots \\
\gamma_4(\xi) =& \underline{X_0} + \overline{X_1}\xi + \overline{X_2}\xi^2 + \underline{X_3}\xi^3 + \underline{X_4}\xi^4 + \overline{X_5}\xi^5 + \overline{X_6}\xi^6 + \cdots
\end{aligned}$$

where the coefficients $\underline{X_i}, \overline{X_i}$ are computed according to (3.160). If these four polynomials are strictly Hurwitz, then the set $\mathcal{I}_n$ is time-invariant strongly robust.

We now make the following remarks.

(2) **Conservatism:** the test in Algorithm 3.3.18 is a conservative test, due to the approximation steps necessary to obtain boxes of systems or interval polynomials and these outer-bounding steps are necessary in order to use the Kharitonov's Theorem in Theorem 3.3.14. As a result, there is the risk that even if the initial box of systems $\mathcal{I}_n$ is time-invariant strongly robust, the test given in Algorithm 3.3.18 does not allow us to draw any conclusion. More general a set in $\mathcal{C}_n$ might be strongly robust whereas it is not enclosable in a strongly robust box of systems. However, the dimensions of the maximal closure box on which the test is actually applied remain proportional to the dimensions of the initial box $\mathcal{I}_n$. Back in an adaptive control framework, if $\mathcal{I}_n$ represents the uncertainty set on the true plant to be controlled, it is crucial to design an input sequence such that this uncertainty set shrinks with time. Under this condition, our test on time-invariant strong robustness is guaranteed to become successful in finite time. This issue is investigated further in Chapter 4.

(2) **Non-orthotopic bounded sets of systems in $\mathcal{C}_n$:** any bounded set Ω can be enclosed in an outer-bounding box of systems in $\mathcal{C}_n$ defined in Definition 3.5. Then, the test presented in Algorithm 3.3.18 can be applied on the obtained outer-bounding box. If this test is positive, i.e., if the outer-bounding set is proved to be time-invariant strongly robust, then Ω is time-strongly robust. Of course, conservatism of the test given in Algorithm 3.3.18 is increased if the bounded set to be tested is not a box of systems.

3.4 Weak strong robustness: the pole placement case

In Section 3.1, we emphasized in Remark 3.1.15 that a given set in $\mathcal{C}_n$ may be strongly robust with respect to a given control objective whilst it is not strongly robust with respect to another control objective. This led to the notion of weak strong robustness defined in Definition 3.1.16. This dependence control objective/strong robustness raises the following questions: given a set $\Omega \subset \mathcal{C}_n$, with respect to what control objectives would Ω be strongly robust? If such control objectives exist, how can they be computed and which one would be the 'closest' to a pre-specified control objective, and what sense can we give to the notion of 'closeness' between two control objectives? Our objective is this section is to investigate these questions in the case of pole placement in some stable real poles. More than a complete study of these questions, this section presents preliminary results which, after a further investigation, could be extended to any type of control objective that satisfies Assumption 2.1.13.

Definition 3.4.1 (Class of pole placements) *Let $\mathcal{F}$ denote the set of pole placements which amounts to place the closed-loop poles in stable poles $(\alpha_1, \cdots, \alpha_{2n-1}) \in (]-1,1[)^{2n-1}$. A pole placement element of $\mathcal{F}$ is said to be admissible for strong robustness for a given set $\Omega \subset \mathcal{C}_n$ if Ω is strongly robust with respect to this pole placement.*

It follows from Definition 3.4.1 that any element in $\mathcal{F}$ can be described completely by a $(2n-1)$-uplet in $(]-1,1[)^{2n-1}$, representing the $2n-1$ desired poles.
Alternatively, any element in $\mathcal{F}$ is also completely described by the $2n-1$ coefficients of the desired closed-loop characteristic polynomial (i.e., the unique monic polynomial which takes its zeros in the desired poles). For instance, consider the pole placement with given desired closed-loop poles $\alpha_1, \cdots, \alpha_{2n-1}$, with $|\alpha_i| < 1$, $\forall i$. Let p denote the corresponding desired closed-loop characteristic polynomial:

$$p(\xi) := p_0 + p_1\xi + \cdots + p_{2n-2}\xi^{2n-2} + \xi^{2n-1} = \prod_{i=1}^{2n-1} (\xi - \alpha_i). \tag{3.161}$$

The considered pole placement can be described by the vector V_1 composed of the desired poles:

$$V_1 = (\alpha_1, \cdots, \alpha_{2n-1})^T \in (]-1,1[)^{2n-1} \tag{3.162}$$

or by the vector V_2 composed of the coefficients of the polynomial p given in (3.161):

$$V_2 = (p_0, \cdots, p_{2n-2}, 1)^T \in \mathbb{R}^{2n+1}. \tag{3.163}$$

3.4.1 Set of pole placements that are admissible for strong robustness

Consider a fixed set of systems $\Omega \subset \mathcal{C}_n$. The question we are asking is the following: what are the pole placements element of $\mathcal{F}$ defined in Definition 3.4.1 with respect to which Ω is strongly robust? In this respect we have the following proposition.

Proposition 3.4.2 (A set of pole placements admissible for strong robustness) *Let $\Omega \subset \mathcal{C}_n$ be a given set of systems. For any pole placement V element in $\mathcal{F}$ defined in Definition 3.4.1, described either by a vector of the form* (3.162) *or* (3.163)*, let f_V denote the control law that assigns with any system $\theta \in \mathcal{C}_n$ its controller $f_V(\theta)$ placing the closed-loop poles in* *the desired poles. Denoting by p the desired characteristic polynomial associated with V according to* (3.161)*, f_V is computed by Ackerman's formula [72]:*

$$f_V(\theta) = -[0 \cdots 0\ 1][B(\theta) \cdots A(\theta)^{2n-1}B(\theta)]p(A(\theta)), \forall \theta \in \mathcal{C}_n, \tag{3.164}$$

where $A(\theta)$ and $B(\theta)$ are given in (3.18) *and* (3.19) *respectively. Now, for any $\theta \in \Omega$, let $\mathcal{B}_{\theta,V}$ denote the ball of systems in $\mathcal{C}_n$ with center θ and radius the complex structured stability radius $r_{\mathbb{C}}A(\theta) + B(\theta)f_V(\theta), B(\theta), I_{2n-1}$ defined in Definition* (3.2.3)*. For any $\theta \in \Omega$, consider the set of pole placements in $\mathcal{F}$ such that for any system $\theta' \in \Omega$, the ball of systems $\mathcal{B}_{\theta,V}$ contains Ω:*

$$\mathcal{F}_\theta = \{V \in \mathcal{F} : \Omega \subset \mathcal{B}_{\theta,V}\} \tag{3.165}$$

Finally define the set $\mathcal{F}_\Omega$ as follows:

$$\mathcal{F}_\Omega = \bigcap_{\theta \in \Omega} \mathcal{F}_\theta. \tag{3.166}$$

If $\mathcal{F}_\Omega \neq \emptyset$, then Ω is strongly robust with respect to any pole placement in $\mathcal{F}_\Omega$, meaning that any pole placement in $\mathcal{F}_\Omega$ is admissible for strong robustness for Ω.

Proof: Suppose that $\mathcal{F}_\Omega \neq \emptyset$. Then, $\forall \theta \in \Omega$ and $\forall V \in \mathcal{F}_\Omega$, $\Omega \subset \mathcal{B}_{\theta,V}$. Hence, $\Omega \subset \bigcap_{\theta \in \Omega} \mathcal{B}_{\theta,V}$. Therefore, it follows from Theorem 3.2.8 that Ω is strongly robust with respect to any $V \in \mathcal{F}_\Omega \neq \emptyset$. Equivalently, $\mathcal{F}_\Omega$ is contained in the set of pole placements in $\mathcal{F}$ that are admissible for strong robustness for Ω. ■

Note that the converse of Proposition 3.4.2 does not necessarily hold, since the largest strongly robust set of systems containing a given system $\theta \in \mathcal{C}_n$ for a given pole placement $V \in \mathcal{F}$ might contain systems that are not element of the ball $\mathcal{B}_{\theta,V}$ defined in Proposition 3.4.2.

The result given in Proposition 3.4.2 provides us with a theoretical way to check whether a given set of systems $\Omega \subset \mathcal{C}_n$ is weakly strongly robust or not. A question that naturally follows this result is the following: how can this test can be practically performed? In the general case of systems of order n, how to compute the sets of systems $\mathcal{B}_{\theta,V}$ defined in Proposition 3.4.2 is not clear yet. However, the first order case is rather simple. Indeed we already saw in Example 3.1.17 that whether a given set $\Omega \subset \mathcal{C}_1$ is weakly strongly robust or not can be checked geometrically. Also, the exact set of pole placements in a stable pole can be computed. When this set is empty, Ω is not weakly strongly robust. On the contrary, when this set is not empty, it depicts exactly the set of pole placement admissible for strong robustness. This idea is illustrated in Figure 3.4.
It should be noted that the discussion addressed in Example 3.1.17 is still valid if the set Ω is not convex nor compact. However, let us focus on the case where Ω is a convex and compact set. Then it is easy to check (geometrically or analytically) that the set of stable poles $\alpha \in]-1,1[$ yielding strong robustness of Ω is convex. This means that if there exists α_1, α_2 in $]-1,1[$ such that $\alpha_1 \leq \alpha_2$, and such that pole placement in α_1 and pole placement in α_2 are admissible for strong robustness for Ω, then Ω is also strongly robust with respect to any pole α such that $\alpha_1 \leq \alpha \leq \alpha_2$. This remark hence suggests that the set of poles $\alpha \in]-1,1[$ yielding strong robustness of Ω may be computed by means of a dichotomy strategy as follows. Suppose that we know a stable pole α_0 such that Ω is strongly robust

with respect to pole placement in α_0.
Fix $\epsilon \in]-1-\alpha_0, 1-\alpha_0[$. Denote $\alpha_1 \to \alpha_0 + \epsilon$.
1. If Ω is strongly robust with respect to pole placement in α_1, then $[\alpha_0, \alpha_1] \subset \mathcal{F}_\Omega$.
2. Otherwise, the upperbound of $\mathcal{F}_\Omega$ is in $[\alpha_0, \alpha_1[$.
By increasing or decreasing ϵ, we would then estimate the lower and upper bounds of the set of stable poles $\alpha \in]-1, 1[$ yielding strong robustness of Ω, this with an arbitrarily good accuracy. Then, since the set of stable poles $\alpha \in]-1, 1[$ yielding strong robustness of Ω is necessarily convex, any pole located between this lower and upper bounds also yields strong robustness for Ω. Such strategy may be generalized to higher order cases, but this problem is still under investigation.

3.4.2 Distance between pole locations

Now, what is interesting is to measure how ”far” a pole placement admissible for strong robustness is located from a given desired control objective. It naturally requires the definition of a notion of ”distance” between two pole placements. Since the the pole placement objective can be characterized in several ways, the notion of distance between two pole placements, however it is defined, depends on the used description. For instance, suppose V, V' to be in the class $\mathcal{F}$ defined in Definition 3.4.1. Then there exist $(\alpha_1, \cdots, \alpha_{2n-1})$ and $(\beta_1, \cdots, \beta_{2n-1})$ in $(]-1,1[)^{2n-1}$ and $(p_0, \cdots, p_{2n-2}, 1)$ and $(q_0, \cdots, q_{2n-2}, 1)$ in $\mathbb{R}^{2n}$ such that

$$p(\xi) := p_0 + p_1\xi + \cdots + p_{2n-2}\xi^{2n-2} + \xi^{2n-1} = \prod_{i=1}^{2n-1} (\xi - \alpha_i), \tag{3.167}$$

$$q(\xi) := q_0 + q_1\xi + \cdots + q_{2n-2}\xi^{2n-2} + \xi^{2n-1} = \prod_{i=1}^{2n-1} (\xi - \beta_i). \tag{3.168}$$

Moreover, V_1 and V_2 can be described by their associated control laws $f, g : \mathcal{C}_n \to \mathbb{R}^{2n-1}$ defined in Assumption 2.1.13 by

$$f(\theta) = -[0 \cdots 0\ 1][B(\theta) \cdots A(\theta)^{2n-1}B(\theta)]p(A(\theta)), \forall\theta \in \mathcal{C}_n, \tag{3.169}$$
$$g(\theta) = -[0 \cdots 0\ 1][B(\theta) \cdots A(\theta)^{2n-1}B(\theta)]q(A(\theta)), \forall\theta \in \mathcal{C}_n, \tag{3.170}$$

where $A(\theta)$ and $B(\theta)$ are given in (3.18) and (3.19) respectively.
Hence we can define the distance between V_1 and V_2 in three different ways. First we can define the distance $d_1(V, V')$ between V_1 and V_2 in terms of distance between the desired poles:

$$d_1(V_1, V_2) = ||(\alpha_1, \cdots, \alpha_{2n-1}) - ||(\beta_0, \cdots, \beta_{2n-1})||. \tag{3.171}$$

Alternatively, we can define the distance $d_2(V_1, V_2)$ between V_1 and V_2 in terms of distance between the desired characteristic polynomials:

$$d_2(V_1, V_2) = ||(p_1, \cdots, p_{2n-2}) - ||(q_0, \cdots, q_{2n-2})||. \tag{3.172}$$

Or, we can define the distance $d_2(V_1, V_2)$ between V_1 and V_2 in terms of distance between the associated control laws:

$$d_3(V_1, V_2) = ||f - g|| := \sup_{\theta \in \Omega} ||f(\theta) - g(\theta)||. \tag{3.173}$$

Remark 3.4.3 The advantage of the distance d_2 in (3.172) is its convenience for computations. However, this metric considers two control objectives V_1, V_2 close to each other as soon as their coefficients p_i, q_i are close to each other whereas from a stability point of view they can be far from each other, since one can be stable and the second unstable. In this respect, the use of the distance d_3 defined in (3.173) might be more judicious since the closeness between two pole placements in the sense of the distance d_3 is somehow more representative of how different the two pole placement will be in terms of the input control action.

3.5 Conclusions

In this chapter we motivated and defined the notion of strong robustness, whilst connecting this notion to classical notions in control theory. One important contribution of this chapter is to present a proof for the existence of strongly robust neighborhood around any system in the considered class of systems. In the given proof, the continuity assumption (see Chapter 2, Assumption 2.1.13) on the map assigning with any system in our class of systems its controller is motivated. Also, various tests resorting to various well known tools in control theory (structured stability radii, infinite or finite LMI's, Kharitonov-like characterization) have been constructed to check whereas specified subsets (balls of systems, polyhedral sets of systems, orthotopic sets of systems) in our class of systems are strongly robust or not with respect to a given control objective. Some of the tests for strong robustness presented in the present chapter are computationally expensive or not tractable and are more a first step rather than a complete solution.

Chapter 4

Set-membership identification for control

In this chapter an input design to guarantee boundedness and decreasing size of the uncertainty set is proposed in the scenario of open-loop identification for adaptive control. Although our aim will be later in this thesis to apply this input design to strong robustness-based adaptive control algorithms, the scope of this chapter can be enlarged to the more general framework of identification for control. The estimated system is a linear time-invariant discrete-time SISO system of known order n with modeling error unknown-but-bounded with a known bound. The key idea in our approach is to consider a $2n$-periodic input sequence and to establish sufficient conditions ensuring boundedness of the uncertainty set in finite time. An iterative input design involving a single design parameter leads then to an uncertainty set of which the volume uniformly decreases with time.

4.1 Introduction

As we saw in Chapter 3, when little information is known on the system to be controlled, the use of a certainty equivalence type of strategy may not be appropriate because asymptotic and global stability of the closed-loop system cannot be guaranteed. In order to resort to certainty equivalence-type control methods, it is hence necessary to first decrease the uncertainty level on the system to be controlled. In this chapter we are concerned with identification of an uncertain linear system with unknown-but-bounded uncertainty with known lower and upper bounds (see Chapter 2). The objective is to collect information of the system of interest until enough is known to allow the use of a certainty equivalence-based control scheme. Since we do not impose any further properties on the modeling error, e.g. statistical properties, common parameter identification schemes such as recursive least squares may not be the appropriate tool. Instead we adopt *set membership identification*, notion which has been extensively studied in the literature in the case of bounded-but-unknown uncertainty ([11], [12], [45], [78], [79] and references therein). Loosely, this amounts to finding a set, the uncertainty set, based on measurements, that contains the true system description.
Because no probabilistic assumptions on the modeling error are imposed, each point in the

uncertainty set is equally likely to represent the true system. Therefore within the uncertainty set there is no natural candidate on the basis of which a controller could be designed. As we saw in Chapter 2, the minimal - but stringent - condition that the uncertainty set has to satisfy so that we can 'safely' start a certainty-equivalence type of strategy is to be strongly robust with respect to the desired control objective. Here the meaning of 'safe' certainty equivalence based control refers to the case where the three drawbacks exposed in Chapter 3 would be non-existent. The question we are investigating in the present chapter is hence the following: how to force identification so as to yield an uncertainty set which is strongly robust in finite time? As previously discussed in Chapter 2, strong robustness is guaranteed to occur provided that the uncertainty set is sufficiently small, i.e., when the radius of the smallest outer-bounding sphere is small. Hence the above question might be re-formulated as follows: how to force identification so as to yield an uncertainty set which becomes arbitrarily small? Since small uncertainty sets may be of interest to any type of control design of an uncertain system, this question may be addressed in a much broader context than just discussions on adaptive control involving strong robustness. Indeed, if the uncertainty set is small enough, it may be expected that a controller designed for some nominal choice will also be useful for every other system in the uncertainty set. A first example of such a nominal choice in the robust control literature is the center of the enclosing sphere of smallest radius, known as *the Chebyshev center* of the uncertainty set ([3], [8], [106], [108]). A second example of such nominal choice is *the analytic center* of the uncertainty set, defined as the point in the uncertainty set which minimizes the logarithmic average output error ([12], [8]). In order to base a control design on such nominal centers for controlling the real unknown system, it is fundamental that the uncertainty set is sufficiently small.

Obviously, the size and shape of the uncertainty set highly depend on the way the system is excited [5]. A good input from an identification point of view is an input which leads to a large amount of information about the real plant, i.e., a small uncertainty set. Such thinking gave rise to the idea of *optimal inputs* [11], [45]: assuming an input structure and an input energy level, one constructs an input so as to minimize a specified measure of the size of the uncertainty set. Unfortunately, such optimal solutions depend on the true unknown system [11], and thus cannot be computed a-priori. Paradoxally, a good approximation of these optimal solutions for a given energy level would require the availability of a good model, hence a small uncertainty. This paradox is typical in adaptive control discussions: the best answer to a problem often depends on the real unknown system to be controlled.

Now, rather than computing an input so as to exactly minimize the size of the uncertainty set, another approach would consist in designing an input such that the largest size of the uncertainty we could possibly obtain is minimized. Such an input would then be optimal in the worst case possible, leading to a *worst-case optimal input*. Let us assume that a bounded uncertainty set has been obtained on the basis of data measurements (this issue will be discussed further in this chapter). Since the true system parameters belong to the uncertainty set at any time, one may compute the new input on the basis of this uncertainty set: this input is the input with the pre-specified energy level which minimizes the worst-case (hence largest) uncertainty set at the next time. A further discussion on this design is postponed until Section 4.3.3.

However, the computation of such an 'optimal input' or 'worst-case optimal input' for a given energy level involves some optimizations that are far from simple and that depend on the input structure. Hence uniqueness of such optimal inputs is not guaranteed, and a comparison of different solutions in terms of suitability to the designer might not be clear. In addition, it must be emphasized that even in the ideal case where such an 'optimal input' or 'worst-case optimal input' of given energy level would be obtained might not be enough for the control purpose. Indeed, to obtain an uncertainty set with a minimized size for a given input energy level does not imply that this size will be small enough to lead to a strongly robust uncertainty set and since it cannot be a-priori known how small the uncertainty set must be in order to be strongly robust, it implies that the designer cannot know a priori to what level the input energy must be set. In the case where the uncertainty would be too large, we should hence re-iterate the design of the input by increasing the allowed input energy level, this until the size becomes sufficiently small. But then, we may wonder whether we really benefit from an iterative design consisting in increasing the energy level, compute the worst case optimal input with this energy level, this until strong robustness is reached, in comparison with the situation where we would fix the input structure and increase its energy level until strong robustness is reached. Obviously the final required energy level using the first method will be smaller than using the second method, however the optimization steps might be of high computational complexity.

These are the reasons why in our approach we do not consider optimal inputs in the sense of inputs that would minimize a measure of the size of the uncertainty set. Instead, we design an input sequence which leads to an uncertainty set that uniformly shrinks with time. At each time instant, the updated input might not be optimal in the sense that the size of the uncertainty set is minimized, but this size is guaranteed to decrease uniformly with time. Hence strong robustness will be achieved in finite time. We assume all along this chapter that a test to check whether this size is indeed small enough or not is available to the designer, meaning that the input sequence that is the object of the present discussion will not be applied infinitely. This problem has been studied in Chapter 3.

Our input design will go along the two following lines. We first select an input structure so as to minimize the number of design parameters, which brings us to select periodic input sequences with period $2n$, denoting by n the assumed system order. Taking the $2n$ input values as parameters, we then derive sufficient conditions so that the uncertainty set becomes arbitrarily "small" for the control purpose, considering two possible measures for size: volume and radius. Our approach differs in two aspects from similar time domain designs reported in the literature. Firstly, we aim explicitly at obtaining small uncertainty sets since small uncertainty sets are necessary for control. This leads inevitably to input sequences of potentially large magnitude. We shall see, however, that due to our construction, the inputs will not grow beyond an unnecessary large threshold. Secondly, the systems that have been studied in the literature regarding optimal periodic input design is restricted to finite impulse response (FIR) systems, that is systems with all poles in the origin [11]. Our class of systems include arbitrary open loop stable input-output systems. We shall see that this is a nontrivial extension. The main reason for that is that, contrary to the FIR case, the input sequence that would minimize the radius of the uncertainty set explicitly depends on the unknown system parameters.

This chapter is organized as follows. In Section 4.2 we formulate the problem statement. Then, in Section 4.3, we focus on a class of periodic input sequences and give sufficient conditions on the input values to guarantee the uncertainty set to be bounded in finite time. However, these conditions depend on the true unknown system, and therefore cannot be explicited a priori. This is the reason why in Section 4.3.3 we focus on the iterative design of an asymptotically periodic input sequence which guarantees that the conditions for boundedness and shrinking volume are satisfied.

4.2 Preliminaries and problem statement

4.2.1 Preliminaries

The uncertain system to be studied is described by linear time-invariant difference equations of the form

$$y(k+1) = (\theta^0)^T \phi(k) + \delta(k+1),\ \forall k, \tag{4.1}$$

where $\phi(k) \in \mathbb{R}^{2n}$ denotes the regressor vector at time k given by:

$$\phi(k) = (-y(k) \cdots - y(k-n+1)\ u(k) \cdots u(k-n+1))^T,\ \forall k,$$

denoting by u, y the input and output sequences respectively, and θ_0 denotes the true unknown parameter vector given by:

$$\theta_0 = (a_{n-1}\ \cdots\ a_0\ b_{n-1}\ \cdots\ b_0)^T \in \mathcal{C}_n \cap \mathcal{S}_n \tag{4.2}$$

where $\mathcal{S}_n$ and $\mathcal{C}_n$ are the set of asymptotically stable systems in $\mathcal{P}_n$ and the set of controllable systems in $\mathcal{P}_n$ respectively as defined in Definition in 2.1.5. Finally, δ denotes the modeling uncertainty and satisfies Assumption 2.1.10, i.e., δ is bounded-but-unknown with known lower and upper bounds $\underline{\delta}$, $\overline{\delta}$ so that

$$\underline{\delta} \leq \delta(k) \leq \overline{\delta},\ \forall k. \tag{4.3}$$

Identification for systems corrupted by deterministic modeling error can be done in many ways. For the case of bounded but otherwise unstructured modeling error a large body of research is devoted to *set-membership identification* (see for instance [12], [79] and the references therein). It consists in computing the *membership set* defined in Chapter 2 as the set of all models that are consistent with the available input-output data $\{\phi(i)\}_{0\leq i\leq k}$:

$$\hat{\mathcal{G}}(k) = \bigcap_{i=0}^{k} \mathcal{G}(i), \tag{4.4}$$

where $\mathcal{G}(k)$, $\forall k$ is given by

$$\mathcal{G}(k) = \{\theta \in \mathbb{R}^{2n} : \underline{\delta} \leq y(k+1) - \theta^T \phi(k) \leq \overline{\delta}\}. \tag{4.5}$$

$\mathcal{G}(k)$ is the hyperstrip in $\mathbb{R}^{2n}$ bounded by the two hyperplanes given by

$$\mathcal{H}_+(k) = \{\theta : y(k+1) = \theta^T \phi(k) + \overline{\delta}\} \text{ and}$$
$$\mathcal{H}_-(k) = \{\theta : y(k+1) = \theta^T \phi(k) + \underline{\delta}\}.$$

Hence, after k measurements, $k \geq 1$, the set $\hat{\mathcal{G}}(k)$ of parameters which are compatible with the assumed model structure (4.1) and the measurements up to time k is given by

$$\mathcal{S}(k) = \mathcal{C}_n \cap \mathcal{S}_n \cap \hat{\mathcal{G}}(k). \tag{4.6}$$

$\mathcal{S}(k)$ defined in (4.6) is the *uncertainty set* at time k. The following properties were established in Chapter 2.

Property 4.2.1

1. *Provided that* $||\phi(k)|| \neq 0$*, the width of the hyperstrip* $\mathcal{G}(k)$ *described in (4.5) is given by*
$$\mathcal{W}(k) = (\overline{\delta} - \underline{\delta})||\phi(k)||^{-1},\ \forall k. \tag{4.7}$$

2. $\forall k \geq 2n-1$*, the intersection formed by the* $2n$ *successive hyperstrips* $\{\mathcal{G}(k-i)\}_{i=0,\cdots,2n-1}$*, is a bounded polytope in* $\mathbb{R}^{2n}$ *if and only if the vectors* $\phi(k-i)$*,* $0 \leq i \leq 2n-1$ *are linearly independent [11], i.e.,*
$$\det(R(k)) \neq 0, \tag{4.8}$$
where the matrix $R(k)$ *is the data matrix defined by*
$$R(k) = [\phi(k-2n+1)\ \cdots\ \phi(k-1)\ \phi(k)] \in \mathbb{R}^{2n\times 2n}. \tag{4.9}$$
Equation (4.8) is an excitation-type condition.

3. *After* N *measurements* $\phi(1), \cdots, \phi(N)$*, supposing that*
$$\max_{k\leq N}\{||\phi(k)||.|\cos(\tilde{\theta}, \phi(k))\ |\} \neq 0, \tag{4.10}$$
the parameter error θ *lies in the ball with center* θ_0 *and radius* $\rho(N)$ *where*
$$\rho(N) = \frac{\overline{\delta} - \underline{\delta}}{\max_{k\leq N}\{||\phi(k)||.\ |\cos(\tilde{\theta}, \phi(k))\ |\}}. \tag{4.11}$$

Clearly, from (4.11), a spanning set of regressors having a large norm with respect to the modeling error levels $\underline{\delta}$, $\overline{\delta}$ and satisfying (4.10) yields a smaller uncertainty set.

4.2.2 Problem formulation

In this chapter, the motivation of the designer is to control the unknown system. By control we here mean the improvement of the performance of the system (4.1). For instance, the control aim could be pole placement. Since this system is unknown, the designer can only rely on an estimated system to design a ”good” controller, provided that this estimate is ”good”. Therefore, we first perform set-membership identification until the uncertainty set is strongly robust. More generally, the objective is to obtain an uncertainty set which is small enough so that a relatively good controller can be obtained on the basis of this uncertainty set. Hence, we emphasize that the identification objective is not to identify the exact description of (4.1). Instead, we assume that the criterion that tells us whether identification can stop so that the effort can switch to control design is satisfied if the uncertainty set is strongly robust in the sense that has been discussed in Chapter 3. Moreover, we here suppose that, given an uncertainty set, it can be measured whether this set is strongly robust or not.

Terminology 4.2.2 The notion of ”size” of a bounded set of systems is taken to be alternatively its radius, defined as the radius of the smallest outer bounding sphere, or its volume, when specified. If the considered set is not bounded, its size is said to be infinite. Finally, we say that a set is ”small” if its size is small.

Our motivation is the following.

Problem Statement 4.2.3 *Consider the system* (4.1)*. Design an input sequence* $\{u(k)\}$ *such that for any initial conditions* $\phi(0)$*, the membership set* $\hat{\mathcal{G}}(k)$ *given in* (4.4) *identified with this input sequence becomes arbitrarily small with time, in the sense defined in Terminology 4.2.2.*

4.3 Membership set estimation with a periodic input

In order to design an input sequence meeting Problem Statement 4.2.3, one reasons as follows. Assume that the two following conditions are satisfied:

i. $\hat{\mathcal{G}}(k)$ is bounded in finite time:

$$\exists k_1 : \det(R(k)) \neq 0,\ \forall k \geq k_1, \tag{4.12}$$

where $R(k)$ is given in (4.9).

ii. The width (4.7) of the strip $\mathcal{G}(k)$ defined in (4.5) uniformly decreases with time:

$$\forall k,\ \exists k' \geq k : \mathcal{W}(k') < \mathcal{W}(k), \tag{4.13}$$

where $\mathcal{W}(k)$ is given in (4.7).

Now remark that as soon as the uncertainty set becomes bounded, it stays bounded at any further time. Moreover the widths of the strips $\mathcal{G}(k)$ are upper-bounds on the dimensions of the uncertainty set $\hat{\mathcal{G}}(k)$ provided that this set is bounded. Thus if i. and ii. are satisfied, the radius of the smallest sphere bounding $\hat{\mathcal{G}}(k)$ becomes arbitrarily small with time. Next, we proceed according the following steps: first we select an input structure, defining our design parameters. Then we establish sufficient conditions on these design parameters so that the conditions i. and ii. are met. Finally, additional conditions are derived so that ii. is met.

4.3.1 Selection of the input structure

In Problem Statement 4.2.3 the structure of the input to be designed is not specified and could a a priori be of any kind. In this section we focus on solutions which would be easily implementable whilst presenting a minimal number of design parameters.
Clearly, in order to have a bounded uncertainty set in the parameter space $\mathbb{R}^{2n}$, $\hat{\mathcal{G}}(k)$ must result from at least $2n$ distinct measurements $\phi(0), \cdots, \phi(2n-1)$ meeting the condition (4.8). Suppose the system (4.1) to be without modeling error ($\delta = 0$). Then, for all $k \geq 2n-1$, (4.8) can be seen as a system of $2n$ linear equations in the input values $u(k-i)$, $i = 0, \cdots, 2n-1$. Thus, $2n$ is in some sense the minimum number of parameters that must be tuned. It is here of relevance to refer to the frequency domain approach: in [102] it is shown that an excitation used for identification of a linear system must effectively contain a minimum number of distinct frequencies, and this minimum number depends on the number of parameters defining the system.
Following this discussion, we consider $2n$-periodic input sequences of the form

$$\tilde{u}(k) = u_{t(k)},\ \forall k \tag{4.14}$$

where the function $t(\cdot) : \mathbb{Z} \longrightarrow \mathbb{Z}_{2n}$ is defined by

$$t(k) = k \mod 2n, \ \forall k \in \mathbb{Z}. \tag{4.15}$$

Because of their simple structure, periodic inputs are easily generated and allow for a simple analysis. For this reason they are typically used for system identification. In a frequency domain approach, such inputs correspond to multi-sine signals ([5], [42], [71], [102]).

4.3.2 Boundedness of the uncertainty set

In this part we consider the class of $2n$-periodic input sequences of type (4.14), taking the $2n$ input values $u_0,\cdots,u_{2n-1}$ as design parameters. The objective is to find sufficient conditions on these parameters so that the uncertainty set $\hat{\mathcal{G}}(k)$ obtained by using the input sequence (4.14) is bounded in finite time. The uncertain system is described by:

$$y(k+1) = (\theta^0)^T \phi(k) + \delta(k+1), \ \phi(0), \tag{4.16}$$

where the regressor vector $\phi(k) \in \mathbb{R}^{2n}$ is given by (4.2). To begin with, the superposition principle allows us to decompose the output sequence y in (4.16) along two components:

$$y(k) = \tilde{y}(k) + y_\delta(k), \ \forall k, \tag{4.17}$$

where $\tilde{y}$ denotes the $2n$-periodic output sequence associated to the uncertainty-free system with $2n$-periodic input sequence $\{\tilde{u}(k)\}$ and y_δ denotes the contribution of the uncertainty δ. We then have:

$$\tilde{y}(k+1) = (\theta^0)^T \tilde{\phi}(k), \ \forall k, \tag{4.18}$$

and

$$\tilde{y}(k) = \tilde{y}(k+2n), \ \forall k, \tag{4.19}$$

with

$$\tilde{\phi}(k) = (-\tilde{y}(k) \cdots - \tilde{y}(k-n+1) \ \tilde{u}(k) \cdots \tilde{u}(k-n+1))^T, \ \forall k. \tag{4.20}$$

The sequence $\tilde{y}$, defined as the unique solution of (4.19), (4.18) is usually referred to as the *steady state output*. In the sequel the symbol ˜ refers to the system in steady-case, i.e., the system with $2n$-periodic input and output signals $\tilde{u}$, $\tilde{y}$. Now, the output component y_δ due to the uncertainty δ is given by

$$y_\delta(k+1) = (\vartheta^0)^T \phi_\delta(k) + \delta(k+1), \ \phi_\delta(0), \tag{4.21}$$

where

$$\vartheta^0 = (a_{n-1} \ \cdots \ a_1 \ a_0)^T \in \mathbb{R}^n, \tag{4.22}$$

and

$$\phi_\delta(k) = (-y_\delta(k) \ \cdots \ y_\delta(k-n+2) \ y_\delta(k-n+1))^T \in \mathbb{R}^n, \forall k. \tag{4.23}$$

In particular, we have

$$\phi(0) = \tilde{\phi}(0) + ((\phi_\delta(0))^T \ u_0 \cdots u_{n+1})^T. \tag{4.24}$$

In order to find sufficient conditions on the parameters $u_0, \cdots, u_{2n-1}$ so that the boundedness condition is satisfied, we now proceed in two steps. We first assume the output sequence is equal to $\tilde{y}$ defined by (4.19), (4.18) and (4.20). This amounts to assume that the system (4.1) to be uncertainty-free ($\delta = 0$) and such that the input and output signals are the steady-state input and output sequence, i.e., are $2n$-periodic. Later we will relax these two assumptions.

System in steady-state without uncertainty

Let us first consider the uncertainty-free system in steady state described by (4.19, 4.18, 4.20) and (4.14). The output sequence $\tilde{y}$ is $2n$-periodic taking values $y_0 = \tilde{y}(0), \cdots, y_{2n-1} = \tilde{y}(2n-1)$ and is hence given by

$$\tilde{y}(k) = y_{t(k)},\ \forall k \tag{4.25}$$

where $t(\cdot)$ is given in (4.15). We easily prove that $y_0, \cdots, y_{2n-1}$ satisfy

$$[y_0\ y_1\ \cdots\ y_{2n-1}]M_1 = [0\ \cdots\ 0\ b_0\ b_1\ \cdots\ b_{n-1}]\ U, \tag{4.26}$$

where $M_1 \in \mathbb{R}^{2n\times 2n}$ is given by

$$M_1 = \begin{bmatrix} 1 & a_{n-1} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 & a_{n-1} & a_{n-2} & & a_0 \\ a_0 & 0 & \ddots & 0 & 1 & a_{n-1} & & a_1 \\ a_1 & a_0 & \ddots & 0 & 0 & 1 & & a_2 \\ \vdots & & \ddots & & & & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{4.27}$$

and $U \in \mathbb{R}^{2n\times 2n}$ is the circulant matrix [40] defined by

$$U = \begin{bmatrix} u_0 & u_1 & u_2 & \cdots & u_{2n-1} \\ u_1 & u_2 & u_3 & \cdots & u_0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ u_{2n-2} & u_{2n-1} & u_0 & \cdots & u_{2n-3} \\ u_{2n-1} & u_0 & u_1 & \cdots & u_{2n-2} \end{bmatrix} \tag{4.28}$$

Let $\tilde{\phi}(k) \in \mathbb{R}^{2n}$ denote the regressor vector with steady-state input/output values given by

$$\tilde{\phi}(k) = (-\tilde{y}(k) \cdots - \tilde{y}(k-n+1)\ \tilde{u}(k) \cdots \tilde{u}(k-n+1))^T.$$

Besides, we denote by $\tilde{R}(k) \in \mathbb{R}^{2n\times 2n}$ the matrix consisting of $2n$ successive regressor vectors given by

$$\tilde{R}(k) = (\tilde{\phi}(k-2n+1)\ \cdots\ \tilde{\phi}(k)). \tag{4.29}$$

Also, we denote by $\tilde{\mathcal{G}}(k)$ the hyperplane computed similarly to (4.4) on the basis of the modeling error-free measurement $\tilde{\phi}(k)$:

$$\tilde{\mathcal{G}}(k) = \{\theta \in \mathbb{R}^{2n} : \tilde{y}(k+1) - \theta^T \tilde{\phi}(k) = 0\} \tag{4.30}$$

and finally $\hat{\tilde{\mathcal{G}}}(k)$ is defined for all $k \geq 0$ as the intersection of the k planes $\tilde{\mathcal{G}}(k)$.

$$\hat{\tilde{\mathcal{G}}}(k) = \bigcap_{i=0}^{k} \tilde{\mathcal{G}}(i). \tag{4.31}$$

Similarly to (4.8), $\forall k \geq 2n-1$, the $2n$ successive hyperplanes $\{\tilde{\mathcal{G}}(k-i)\}_{i=0,\cdots,2n-1}$ are linearly independent in $\mathbb{R}^{2n}$ if and only if

$$\det(\tilde{R}(k)) \neq 0. \tag{4.32}$$

Let us introduce Q_0 to be the following matrix:

$$Q_0 = \begin{bmatrix} -y_{n-1} & -y_n & y_{n+1} & \cdots & -y_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ -y_1 & -y_2 & -y_3 & \cdots & -y_0 \\ -y_0 & -y_1 & -y_2 & \cdots & -y_{2n-1} \\ u_{n-1} & u_n & u_{n+1} & \cdots & u_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ u_1 & u_2 & u_3 & \cdots & u_0 \\ u_0 & u_1 & u_2 & \cdots & u_{2n-1} \end{bmatrix} \in \mathbb{R}^{2n\times 2n} \tag{4.33}$$

Definition 4.3.1 *Given any matrix T, we call row-permutation of T any matrix obtained by permutations of the rows of T. Similarly, we call column-permutation of T any matrix obtained by permutations of the columns of T. And we call permutation of T any matrix obtained by row-permutations and column-permutations of T. The relation "is a permutation of" is denoted by $\sim$.*

The following can be easily verified:

$$\begin{aligned} \tilde{R}(k) &= Q_0, \forall k = n-2 \mod 2n, k \geq 2n-1, \\ \tilde{R}(k) &\sim Q_0, \forall k \neq n-2 \mod 2n, k \geq 2n-1. \end{aligned} \tag{4.34}$$

Hence,

$$\forall k \geq 2n-1,\ \det(\tilde{R}(k)) \neq 0 \Longleftrightarrow \det(Q_0) \neq 0. \tag{4.35}$$

Moreover, the following theorem can easily be verified.

Theorem 4.3.2 *M_1, Q_0 and U satisfy the equation:*

$$Q_0 M_1 = M_2 U \tag{4.36}$$

where $M_2 \in \mathbb{R}^{2n\times 2n}$ is given by

$$M_2 = \begin{bmatrix} -b_1 & -b_2 & \cdots & & -b_{n-1} & 0 & 0 & 0 & \cdots & 0 & -b_0 \\ -b_2 & -b_3 & \cdots & -b_{n-1} & 0 & 0 & 0 & 0 & \cdots & -b_0 & -b_1 \\ \vdots & & \vdots & & \vdots & & & & & \vdots & \vdots \\ -b_{n-1} & 0 & 0 & & 0 & 0 & 0 & -b_0 & \cdots & -b_{n-3} & -b_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & -b_0 & -b_1 & \cdots & -b_{n-2} & -b_{n-1} \\ a_1 & a_2 & \cdots & & a_{n-1} & 1 & 0 & 0 & \cdots & 0 & a_0 \\ a_2 & a_3 & \cdots & a_{n-1} & 1 & 0 & 0 & 0 & \cdots & a_0 & a_1 \\ \vdots & & \vdots & & \vdots & & & & & \vdots & \vdots \\ a_{n-1} & 1 & 0 & & 0 & 0 & 0 & a_0 & \cdots & a_{n-3} & a_{n-2} \\ 1 & 0 & 0 & \cdots & 0 & 0 & a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{bmatrix} \tag{4.37}$$

Then, in [40], the following theorem is proved.

Theorem 4.3.3 *Let $M(x_0, x_1, \cdots, x_{N-1})$ denote the $N \times N$ circulant matrix with entries $x_0, \cdots, x_{N-1}$ in $\mathbb{C}$:*

$$M(x_0, \cdots, x_{N-1}) = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{N-1} \\ x_1 & x_2 & x_3 & \cdots & x_0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_{N-2} & x_{N-1} & x_0 & \cdots & x_{N-3} \\ x_{N-1} & x_0 & x_1 & \cdots & x_{N-2} \end{bmatrix} \tag{4.38}$$

$M(x_0, \cdots, x_{N-1})$ *has full row rank if and only if* $\gcd(\sum_{i=0}^{N-1} x_i \xi^i, \xi^N - 1) = 1$.

From Theorem 4.3.3 we derive the following corollary.

Corollary 4.3.4 *The matrix M_1 defined in* (4.27) *is invertible.*

Proof: Using Definition 4.3.1, one can easily show that $M_1 \sim M'$, where

$$M' = M(1, a_{n-1}, a_{n-2}, \cdots, a_0, 0, 0, \cdots, 0) \in \mathbb{R}^{2n \times 2n}, \tag{4.39}$$

and where the matrix $M(.)$ is defined in (4.38). Hence $\det(M_1) = \det(M')$. Since the true system is asymptotically stable, i.e., $\theta^0 \in \mathcal{S}_n$, it follows from 2.1.7 that $\mathcal{A}(\xi) = \xi^n + \sum_{i=0}^{n-1} a_i \xi^i$ has no zero on the unit circle. Thus $1 + a_{n-1}\xi^1 + \cdots + a_0 \xi^n = \xi^n \mathcal{A}(\xi^{-1})$ has no zero on the unit circle, i.e., is co-prime with $\xi^{2n} - 1$. From Theorem 4.3.3, this implies that $\det(M') \neq 0$ and therefore $\det(M_1) \neq 0$. This concludes the proof of Theorem 4.3.4. ■

Now we have the following Theorem.

Theorem 4.3.5 *Under the condition* (4.2)*, the following are equivalent:*

$$\text{i. } \hat{\tilde{\mathcal{G}}}(k) \text{ is bounded}, \ \forall k \geq 2n - 1 \tag{4.40}$$

$$\text{ii. } \det(\tilde{R}(k)) \neq 0, \ \forall k \geq 2n - 1 \tag{4.41}$$

$$\text{iii. } \det(U) \neq 0 \tag{4.42}$$

$$\text{iv. } \gcd(\textstyle\sum_{i=0}^{2n-1} u_i \xi^i, \xi^{2n} - 1) = 1 \tag{4.43}$$

Proof:
i. ⇔ ii. Due to the $2n$ periodicity of $\tilde{\phi}$, we have: $\hat{\tilde{\mathcal{G}}}(k) = \bigcap_{i=0}^{2n-1} \tilde{\mathcal{G}}(k-i)$, $\forall k \geq 2n-1$. We conclude remarking that $\det(\tilde{R}(k)) \neq 0 \Leftrightarrow \bigcap_{i=0}^{2n-1} \tilde{\mathcal{G}}(k-i)$ is bounded, $\forall k \geq 2n-1$.
ii. ⇔ iii. Corollary 4.3.4 implies that M_1 is invertible. It follows from Theorem 4.3.2 that $\det(\tilde{R}(k)) = \det(M_2 U M_1^{-1})$, $\forall k \geq 2n-1$. Now, remark that the matrix $M_2 \in \mathbb{R}^{2n \times 2n}$ given in (4.37) satisfies $M_2 \sim S(\mathcal{A}, \mathcal{B})$ where $S(\mathcal{A}, \mathcal{B})$ is the Sylvester matrix associated with the polynomials $\mathcal{A}(\xi) = \xi^n + \sum_{k=0}^{n-1} a_k \xi^k$, $\mathcal{B}(\xi) = \sum_{k=0}^{n-1} b_k \xi^k$ [93]. Hence, under the controllability assumption in (4.2), $S(\mathcal{A}, \mathcal{B})$ is invertible, implying that the matrix M_2 in (4.37) is invertible. Since M_1 is also invertible (see Corollary 4.3.4), then $\det(\tilde{R}(k) \neq 0 \Leftrightarrow \det(U) \neq 0$, $\forall k \geq 2n-1$.
iii. ⇔ iv. directly follows from Theorem 4.3.3. ■

Remark 4.3.6 If $\det(M_2) = 0$, then $\det(\tilde{R}(k)) = 0$, $\forall k \geq 2n-1$, $\hat{\tilde{\mathcal{G}}}(k)$ is thus never bounded. Hence the controllability assumption in (4.2) is crucial. If $\det(M_1) = 0$, i.e., if the system to be controlled is not asymptotically stable, no conclusion can be drawn about the boundedness of the set $\hat{\tilde{\mathcal{G}}}(k)$.

Remark 4.3.7
(1) An interesting problem is the following: how to choose the input sequence so that any $2n$ successive hyperplanes $\tilde{\mathcal{G}}(k-i)$, $i = 0, \cdots, 2n-1$ are as orthogonal as possible to each other? Remark that in the case where $\mathcal{K}(\tilde{R}(k)) = 1$, where $\mathcal{K}(\tilde{R}(k))$ denotes the condition number of the matrix $\tilde{R}(k)$ defined in (4.29), then $\tilde{R}(k)$ is an orthogonal matrix, i.e., the planes $\{\tilde{\mathcal{G}}(k-i)\}_{i=0,\cdots,2n-1}$ are exactly perpendicular to each other. Moreover, in the case of a condition number close to 1, the sensitivity of the boundedness condition (4.42) with respect to modeling errors would be minimized. Hence, we consider the optimization problem which consists in minimizing $\mathcal{K}(\tilde{R}(k)) = \mathcal{K}(M_2 U M_1^{-1})$ over the set of matrices U, where U, M_1 and M_2 are defined in (4.28), (4.27) and (4.37). However, we see that because $\mathcal{K}(M_2)$ and $\mathcal{K}(M_1)$ are unknown, to minimize $\mathcal{K}(\tilde{R}(k))$ over the class of matrices U defined in (4.28) is an ill-posed problem. Indeed, a matrix U which would minimize $\mathcal{K}(M_2 U M_1^{-1})$ for given matrices M_1, M_2 does not necessarily minimize $\mathcal{K}(M_2' U M_1'^{-1})$ for another choice M_1', M_2' of the form (4.27), (4.37). This is easily checked in the case of first order systems, i.e., when $n = 1$. To illustrate this result, let us consider the two choices of set of matrices (M_1, M_2), (M_1', M_2'), defined by $a_0 = 0.5, b_0 = 5$ and $a_0' = 0.9, b_0' = 0.5$ respectively. They are hence given by

$$M_1 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \; M_2 = \begin{bmatrix} 0 & 5 \\ 1 & 0.5 \end{bmatrix} \tag{4.44}$$

$$M_1' = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix} \; M_2' = \begin{bmatrix} 0 & 0.5 \\ 1 & 0.9 \end{bmatrix} \tag{4.45}$$

and we have

$$U = \begin{bmatrix} u_0 & u_1 \\ u_1 & u_0 \end{bmatrix}. \tag{4.46}$$

In this first order case, we have that the 2-periodic input sequence $\{u_0, u_1, u_0, u_1, \cdots\}$ satisfies (4.8) if and only if:

$$u_0^2 \neq u_1^2, \tag{4.47}$$

hence at least one of the two variables u_0, u_1 has to be not zero. Without loss of generality, let us suppose that $u_1 \neq 0$. Then we can write $U = u_1.U'$ where U' is the matrix defined by:

$$U' = \begin{bmatrix} \frac{u_0}{u_1} & 1 \\ 1 & \frac{u_0}{u_1} \end{bmatrix}. \tag{4.48}$$

And clearly we have that

$$\mathcal{K}(M_2 U M_1^{-1}) = \mathcal{K}(M_2 U' M_1^{-1}), \; \mathcal{K}(M_2' U (M_1')^{-1}) = \mathcal{K}(M_2' U' (M_1')^{-1}). \tag{4.49}$$

Hence, to study $\mathcal{K}(M_2 U M_1^{-1})$ and $\mathcal{K}(M_2' U (M_1')^{-1})$ as functions of the variables u_0, u_1 is equivalent to study $\mathcal{K}(M_2 U M_1^{-1})$ and $\mathcal{K}(M_2' U (M_1')^{-1})$ as functions of one variable $\frac{u_0}{u_1}$. For this reason, we can suppose that $u_1 = 1$ and study $\mathcal{K}(M_2UM_1^{-1})$ and $\mathcal{K}(M_2'U(M_1')^{-1})$ as functions of one variable u_0. In Figure 4.1, we represent the condition numbers $\mathcal{K}(M_2UM_1^{-1})$ and $\mathcal{K}(M_2'U(M_1')^{-1})$ for $u_1 = 1$, $a_0 = .5, b_0 = 5$ and $a_0' = .9, b_0' = .5$, as functions of $u_0 \in \mathbb{R}\backslash\{1, -1\}$. If $|u_0| = 1$, i.e., if the excitation condition (4.8) is not satisfied, we have $\mathcal{K}(M_2UM_1^{-1}) = \mathcal{K}(M_2'U(M_1')^{-1}) = \infty$, i.e., the matrix products $M_2UM_1^{-1}$ and $M_2'U(M_1')^{-1}$ are ill-conditioned. These plots show that when $\mathcal{K}(M_2UM_1^{-1})$ is minimized, $\mathcal{K}(M_2'U(M_1')^{-1})$ is not minimized and vice-versa. Hence, the input value u_0 that minimizes the condition number of the product $M_2UM_1^{-1})$ depends on the choice of M_1, M_2 of the form (4.27), (4.37). Since the true system parameters a_0, b_0 are unknown, it is therefore not possible to minimize $\mathcal{K}(M_2UM_1^{-1})$ a priori.

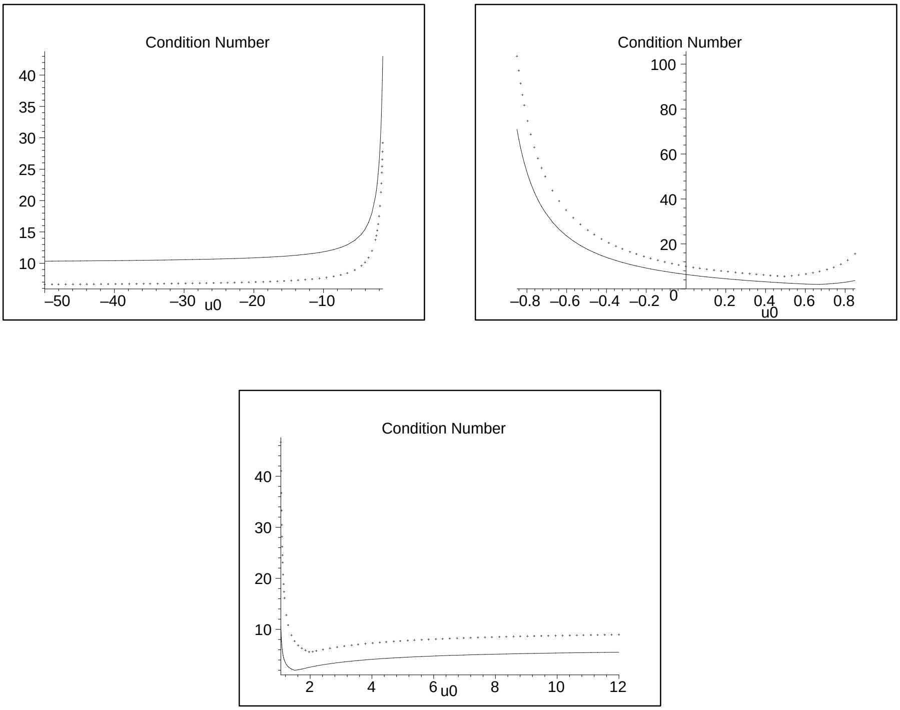


Figure 4.1: $\mathcal{K}(M_2UM_1^{-1})$ for $(a_0, b_0) = (.5, 5)$ (dotted line), $(a_0', b_0') = (.9, .5)$ (solid line).

For higher order systems this effect is likely to be even stronger, but this is difficult to analyze. Monte Carlo simulations, however, confirm this intuition and show that for different parameter sets, the condition number reaches its minimum values in different values of the input values.

(2) Although the lack of knowledge of the matrices M_1, M_2 prevents us from minimizing $\mathcal{K}(M_2UM_1^{-1})$ over the class of matrices U of the type (4.28), it is worth to note that

$$\mathcal{K}(M_2UM_1^{-1}) \leq \mathcal{K}(M_2).\mathcal{K}(U).\mathcal{K}(M_1^{-1}) = \mathcal{K}(M_2).\mathcal{K}(U).\mathcal{K}(M_1). \tag{4.50}$$

Now, since M_1 and M_2 are fixed (but unknown), the upper bound of $\mathcal{K}(M_2 U M_1^{-1})$ on the right hand side of (4.50) is minimal when $\mathcal{K}(U) = 1$. We saw earlier that M_1 and M_2 are invertible, hence the term $\mathcal{K}(M_2).\mathcal{K}(U).\mathcal{K}(M_1)$ is finite provided that U is invertible. Clearly, the situation where $\mathcal{K}(U) = 1$ by no means implies that $\mathcal{K}(M_2 U M_1^{-1})$ is minimal. However, for any matrix U of the form (4.28), if $\det(U) \neq 0$, then it follows from Theorem 4.3.5 that $\hat{\hat{\mathcal{G}}}(k)$ is bounded $\forall k \geq 2n-1$. A nice property of this result is that it is independent of the unknown system (4.1). Following this discussion, we now focus on the parameterization of periodic sequences that are solutions of $\mathcal{K}(U) = 1$. We have the following result.

Result 4.3.8 *The two following statements are equivalent.*

i. $\mathcal{K}(U) = 1$ *where* U *is given in* (4.28).

ii. $u_0, \cdots, u_{2n-1}$ *meet the* $n+1$ *following conditions:*

$$\exists i \in [0, 2n-1] \text{ such that } u_i \neq 0 \text{ and } \sum_{p=0}^{2n-1} u_{t(p)} u_{t(p+j)} = 0,\ \forall j \in [1, n], \tag{4.51}$$

where $t(k) = k \mod 2n$, $\forall k \in \mathbb{N}$.

Proof: Denote by ω_k, $k = 0, \cdots, 2n-1$ the unit roots given by

$$\omega_k = e^{i\frac{2\Pi k}{2n}}. \tag{4.52}$$

It is shown in [65] that for any sequence $\{x_i\}_{i=0,\cdots 2n-1} \in \mathbb{C}^{2n}$, the eigenvalues of the circulant matrix $M(x_0, \cdots, x_{2n-1})$ of the form (4.38) are given by:

$$\lambda_k(M(x_0, \cdots, x_{2n-1})) = (\sum_{m=0}^{2n-1} \omega_k^m x_m)^{\frac{1}{2}} \in \mathbb{R}_+,\ k = 0, \cdots, 2n-1. \tag{4.53}$$

Now, we have that the singular values of U given in (4.28) are given by

$$\sigma_k(U) = \sqrt{\lambda_k(U^T U)},\ \forall k = 0, \cdots, 2n-1. \tag{4.54}$$

Now, we easily check that the matrix $U^T U$ is the circulant matrix satisfying

$$U^T U = M(x_0, \cdots, x_{2n-1}), \tag{4.55}$$

where $x_0, \cdots, x_{2n-1}$ are given by

$$x_m = \sum_{p=0}^{2n-1} u_{t(p)} u_{t(p+m)},\ m = 0, \cdots, 2n-1. \tag{4.56}$$

Hence, using (4.53) and (4.54), the singular values of U are given by

$$\sigma_k(U) = (\sum_{m=0}^{2n-1} \omega_k^m x_m)^{\frac{1}{2}} \in \mathbb{R}_+,\ k = 0, \cdots, 2n-1, \tag{4.57}$$

where $x_0, \cdots, x_{2n-1}$ are given in (4.56). Using the Frobenius norm for the matrix norm, we have: $\mathcal{K}(U)=\overline{\sigma}/\underline{\sigma}$ where $\overline{\sigma}$, $\underline{\sigma}$ denote the largest and the smallest singular values of U respectively. Hence $\mathcal{K}(U)=1$ if and only if $\sigma_k(U)=\sigma_p(U)$, $\forall(k,p)$ and $\sigma_k(U)\neq 0$, $\forall k$. For all $p=0,\cdots,2n-1$, let $P(\xi), Q_p(\xi) \in \mathbb{C}[\xi]$ denote the polynomials defined by:

$$P(\xi) = \sum_{m=0}^{2n-1} x_m \xi^m, \tag{4.58}$$

$$Q_p(\xi) = P(\xi) - P(\omega_p \xi) = \sum_{m=1}^{2n-1} (1-\omega_p^m) x_m \xi^m. \tag{4.59}$$

Using (4.58), $\mathcal{K}(U)=1$ if and only if:

$$P(\omega_k) = P(\omega_p), \forall k, p = 0, \cdots, 2n-1. \tag{4.60}$$

Then, notice that:

$$\omega_k = \omega_p.\omega_{k-p}, \forall k, p = 0, \cdots, 2n-1. \tag{4.61}$$

Hence, using (4.59), (4.60) and (4.61), we obtain that $\mathcal{K}(U) = 1$ if and only if $\forall k, p = 0, \cdots, 2n-1$ the following is satisfied:

$$\begin{aligned} Q_p(\omega_{k-p}) &= P(\omega_{k-p}) - P(\omega_p.\omega_{k-p}) \\ &= P(\omega_{k-p}) - P(\omega_k) = 0. \end{aligned} \tag{4.62}$$

(4.62) implies that $\forall p = 0, \cdots, 2n-1$, the polynomial $Q_p(\xi)$ of degree $2n-1$ has $2n$ distinct zeros $\{\omega_j\}_{j=0,\cdots,2n-1}$. Therefore, $\forall p = 0, \cdots, 2n-1$, the polynomial $Q_p(\xi)$ is the zero polynomial, which implies that $x_m = 0$, $\forall m = 1, \cdots, 2n-1$. Hence, $\mathcal{K}(U)=1$ is satisfied if and only if $\{x_m, m = 0, \cdots, 2n-1\} = \{x_0, 0, \cdots, 0\}$, with $x_0 \neq 0$, which is equivalent to say that (4.51) is satisfied. ■

The following corollary follows from Result 4.3.8.

Corollary 4.3.9 *Consider $2n$ real-valued numbers $u_0, \cdots, u_{2n-1}$ meeting the $n+1$ following conditions:*

$$\exists i \in [0, 2n-1] \text{ such that } u_i \neq 0 \text{ and } \sum_{p=0}^{2n-1} u_{t(p)} u_{t(p+j)} = 0,\ \forall j \in [1, n], \tag{4.63}$$

where $t(k) = k \mod 2n$, $\forall k \in \mathbb{N}$. Let $\tilde{u}$ denote the $2n$-periodic input sequence defined by

$$\tilde{u}(k') = u_k,\ \forall k' = k \mod 2n,\ \forall k. \tag{4.64}$$

Then the identification of the uncertainty-free system in steady-state given by (4.18) *using the input $\tilde{u}$ would be such that $\hat{\hat{\mathcal{G}}}(k) = \{\theta^0\}$, $\forall k \geq 2n-1$.*

Proof: Suppose that the input sequence is given by (4.64) where $u_0, \cdots, u_{2n-1}$ satisfy (4.63). It follows from Result 4.3.9 that $\mathcal{K}(U) = 1$, where U is given in (4.28). Hence, Theorem 4.3.5 means that $\hat{\tilde{\mathcal{G}}}(k)$ is bounded for any $k \geq 2n-1$. Since $\delta = 0$, the strips $\tilde{\mathcal{G}}(k)$, $\forall k$, are in fact reduced to hyperplanes. Hence boundedness of the intersection $\hat{\tilde{\mathcal{G}}}(2n-1)$ implies that $\hat{\tilde{\mathcal{G}}}(2n-1)$ is reduced to a point set. Since $\theta^0 \in \hat{\tilde{\mathcal{G}}}(k)$ for all k, this implies that $\hat{\tilde{\mathcal{G}}}(2n-1) = \{\theta^0\}$. And therefore $\hat{\tilde{\mathcal{G}}}(k) = \{\theta^0\}$, $\forall k \geq 2n-1$. ■

Remark 4.3.10 Any $2n$-periodic impulse sequence of the form

$$\{\tilde{u}(k), k = 0, \cdots, 2n-1\} = \{0, \cdots, 0, u, 0, \cdots, 0\}, u \neq 0 \tag{4.65}$$

satisfies (4.63). Note, however, that the $2n$-periodic sequences which satisfy (4.63) are not necessarily impulse sequences. For instance, for $n = 2$, the 3-periodic sequences of the form

$$\{\tilde{u}(k), k = 0, \cdots, 3\} = \{u_0, u_1, \frac{u_1^2}{u_0}, -u_1\}, u_0 \neq 0 \tag{4.66}$$

satisfies (4.51) and are not necessarily impulse sequences of the type (4.65).

System with modeling error and arbitrary initial conditions

We now suppose that the modeling uncertainty δ in (4.1) is non zero and that the system (4.1) starts in any initial state. Hence the system is now taken to be given by (4.1, 4.2, 4.2, 4.3) and (4.14). As it has been previously discussed, we have:

$$y(k) = \tilde{y}(k) + y_\delta(k), \forall k, \tag{4.67}$$

where $\tilde{y}(k)$ has been previously studied and is given by (4.25), (4.15) and (4.26). Let us now focus on the output component y_δ due to the uncertainty on the system to be studied. It satisfies the following equations:

$$y_\delta(k+1) = \vartheta_0^T \phi_\delta(k) + \delta(k+1),\ \phi_\delta(0), \tag{4.68}$$

where the parameter vector ϑ_0 and the regressor vector $\phi_\delta(k)$ are given by

$$\vartheta_0 = (a_{n-1}\ \cdots\ a_1\ a_0)^T \tag{4.69}$$

$$\phi_\delta(k) = (-y_\delta(k) \cdots - y_\delta(k-n+2)\ \ -y_\delta(k-n+1))^T,\ \forall k \geq 1. \tag{4.70}$$

It can be shown that

$$|y_\delta(k)| \leq ||A_0||^k |\phi_\delta(0)| + \max(|\overline{\delta}|, |\underline{\delta}|) \sum_{j=0}^{k-1} ||A_0||^j,\ \forall k \geq 1, \tag{4.71}$$

where $||.||$ denotes the Frobenius norm in $\mathbb{R}^{n\times n}$ and $A_0 \in \mathbb{R}^{n\times n}$ and $B_0 \in \mathbb{R}^{n\times 1}$ are given by:

$$A_0 = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} B_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Since the true system parameter vector is in $\mathcal{S}_n$, i.e., the true system is asymptotically stable, then the polynomial $\mathcal{A}(\xi) = \xi^n + \sum_{i=0}^{n-1} a_i \xi^i$ is strictly Schur. We easily check that $\det(A_0 - \xi I_n) = \mathcal{A}(\xi)$ hence A_0 is Schur stable. Therefore the expression in the right side in (4.71) is bounded by a constant depending on the matrix A_0, the initial condition $\phi_\delta(0)$ and the modeling error level $\overline{\delta}$. Denoting by $\overline{y_\delta}$ this constant, we obtain that the output component y_δ is bounded by a constant depending on $\overline{\delta}$ and the initial condition $\phi_\delta(0)$:

$$|y_\delta(k)| \leq \overline{y_\delta},\ \forall k \geq 0. \tag{4.72}$$

We have the following theorem:

Theorem 4.3.11 (Bounded uncertainty set) *For any $\gamma \in \mathbb{R}_+$, define the $2n$-periodic input sequence of the form*

$$u_\gamma(k) = \gamma.\tilde{u}(k), \forall k \tag{4.73}$$

where $\tilde{u}$ is a $2n$-periodic sequence of the type (4.14) *such that its values $u_0, \cdots, u_{2n-1}$ satisfy* (4.42). *Then there exists $\gamma > 0$ such that the uncertainty set $\hat{\mathcal{G}}(k)$ associated with the system (4.1) excited by the input* (4.73) *for this value of γ is bounded for any $k \geq 2n-1$.*

Proof With any $\gamma > 0$, associate the input sequence $\{u_\gamma(k)\}$ described by (4.73) where the values u_i satisfy (4.42). Now remark that for any $k \geq 2n-1$, $R(k) = \tilde{R}(k) + \Delta(k)$ where $\tilde{R}(k)$ is defined in (4.29) and $\Delta(k) \in \mathbb{R}^{2n\times 2n}$ is given by:

$$\Delta(k) = \begin{bmatrix} y_\delta(k-2n+1) & y_\delta(k-2n+2) & \cdots & y_\delta(k) \\ \vdots & \vdots & \cdots & \vdots \\ y_\delta(k-3n+3) & y_\delta(k-3n+4) & \cdots & y_\delta(k-n+2) \\ y_\delta(k-3n+2) & y_\delta(k-3n+3) & \cdots & y_\delta(k-n+1) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

where $|y_\delta(i)| \leq \overline{y_\delta}, \forall i$, with $\overline{y_\delta}$ deduced from (4.71). Hence we have:

$$||\Delta(k)||_\infty \leq \overline{y_\delta}, \forall k \geq 2n-1. \tag{4.74}$$

Now, define the function $\Lambda : \mathbb{R}^{2n\times 2n} \to \mathbb{R}_+$ by:

$$\Lambda(\Delta) = \det(Q_0 + \Delta) \tag{4.75}$$

where Q_0 is given in (4.33). Using Theorem 4.3.5 and (4.42), $\Lambda(0) \neq 0$. Moreover, Λ is continuous. Hence $\exists \delta_1 > 0$ such that: if $||\Delta||_\infty < \delta_1$, then $\Lambda(\Delta) \neq 0$. Hence, $\forall k \geq 2n-1$, $\forall \gamma > 0$, if $||\Delta||_\infty/\gamma < \delta_1$, then $\Lambda_k(1/\gamma.\Delta) \neq 0$. For all $k \geq 2n-1$, we have:

$$\Lambda(\Delta(k)/\gamma) = \det(Q_0 + \Delta(k)/\gamma) = \gamma^{-2n} \det(\gamma Q_0 + \Delta(k)),$$

where the matrix $\Delta(k)$ is defined in (4.74). Moreover, $\forall k = n-2 \mod 2n-1$, $k \geq 2n-1$, we have that $\tilde{R}(k) = \gamma.Q_0$, hence, $\Lambda(\Delta(k)/\gamma) = \gamma^{-2n}\det(R(k))$ with $R(k)$ defined by (4.9). We hence proved that $\forall k = n-2 \mod 2n-1$, $k \geq 2n-1$, $\exists \delta_1 > 0$ such that $\forall \gamma$, if $||\Delta(k)||_\infty/\gamma \leq \delta_1$, then $\gamma^{-2n}\det(R(k)) \neq 0$, i.e., $\det(R(k)) \neq 0$. Take $\gamma_0 = \frac{\overline{y_\delta}}{\delta_1}$. Then, $\forall k = n-2 \mod 2n-1$, $k \geq 2n-1$, if $||\Delta(k)||_\infty \leq \overline{y_\delta}$, identifying the system (4.1) using the input defined by (4.73) with any value $\gamma \geq \gamma_0$ guarantees that $\det(R(k)) \neq 0$, i.e., $\hat{\mathcal{G}}(k)$ is bounded. The case where $k \neq n-2 \mod 2n$, $k \geq 2n-1$ is treated similarly, replacing Q_0 by a matrix obtained after a finite number of cyclic permutations on the columns of Q_0. This concludes the proof of Theorem 4.3.11. ■

4.3.3 Arbitrarily small unfalsified set

We now use the results previously established to complete the design of an input sequence yielding an arbitrarily small uncertainty set. First we prove the existence of such an input sequence and later we see how this input can be explicitely designed.

Input sequence design for an arbitrarily small uncertainty set: existence

Many discussions involving the size of the uncertainty set and the design of optimal inputs that would minimize such size can be found in the literature [1], [5], [11], [45].
Suppose the system (4.1) to be excited by a $2n$-periodic input of the form (4.14) where $u_0, \cdots, u_{2n-1}$ satisfy $|u_i| \leq \Gamma$ for a fixed $\Gamma > 0$. Suppose moreover that the output is in steady-state, i.e., $2n$-periodic. Let Ω_m denote the intersection given by

$$\Omega_m = \bigcap_{k=m}^{m+2n-1} \mathcal{G}(k),\ \forall m, \tag{4.76}$$

where $\mathcal{G}(k)$ is given in (4.5). In [11], the authors show that when the system is in steady state, the volume Vol(Ω_m) and the diameter Dia(Ω_m) of Ω_m satisfy:

$$\text{Vol}(\Omega_m) \leq \frac{(\overline{\delta}-\underline{\delta})^{2n}}{|\det(\tilde{R}(m))|}, \quad \forall m \geq 2n-1, \tag{4.77}$$

$$\text{Dia}(\Omega_m) \leq (\overline{\delta}-\underline{\delta})||\tilde{R}(m)^{-1}||\sqrt{2n}, \quad \forall m \geq 2n-1, \tag{4.78}$$

respectively, provided that (4.42) holds. The matrix $\tilde{R}(m)$ is defined in (4.29). Moreover, it is shown that these bounds are the tightest bounds that can be obtained if we have no information on the modeling uncertainty except that it is lower and upper bounded by $\underline{\delta}, \overline{\delta}$. For FIR systems in steady state, as it is shown in [11], minimization of the upperbounds in (4.77) and (4.78) is equivalent to the minimization of $\det(U)$ and $||U^{-1}||$ respectively. In both cases the minimizing input sequence does not depend on the unknown system parameters. In our framework, however, $\tilde{R}(m) \sim M_2 U M_1^{-1}$ where M_1, M_2 are given in (4.27), (4.37). Hence the upperbound on the volume given in (4.77) is minimized for the input values such that

$$\{u_i\} = \arg\max_{|u_i|\leq\Gamma} |\det(M_2 U M_1^{-1})| = \arg\max_{|u_i|\leq\Gamma} |\det(U)|, \tag{4.79}$$

and the upperbound on the diameter given in (4.78) is minimized for the input values such that

$$\{u_i\} = \arg\min_{|u_i|\leq\Gamma} ||M_1 U^{-1} M_2^{-1}||. \tag{4.80}$$

It is shown in [11] that minimizing the bound on the volume in (4.77) does not necessarily lead to inputs that minimize the bound on the diameter given in (4.78), and vice-versa. Indeed, in the case where the uncertainty set would have almost all its dimensions very small but one of these dimensions very large could lead to a small volume but a large radius. For our control purpose, small in the sense of radius is more appropriate than small in the sense of volume.
It is interesting to remark that even in our case, the input values satisfying (4.79) do not depend on the real system parameters. However, due to the structure of the system (4.1), the input values satisfying (4.80) do depend on these unknown parameters. And also, even in the case the radius of the uncertainty set is minimized, its value depends on the parameter vector so it might happen that this radius will still be too large for our control purpose. Hence, from a practical point of view, it is not clear whether it makes sense to design an input sequence with the idea to minimize the volume or the radius of the uncertainty set, since such input depends on the unknown system.
Alternatively, one may reason as follows: since the true system parameters lie in the uncertainty set at any time, and since this uncertainty set is bounded, it is theoretically possible to compute the input values $u_0, \cdots, u_{2n-1}$ which would minimize the largest radius we could possibly obtain at the next time. At time $k+1$, these values u_i^{k+1} are solutions of the 'worst-case' minimization problem:

$$\{u_i^{k+1}\} = \arg \min_{|u_i| \leq \Gamma} \{\arg \max_{\theta \in \hat{\tilde{\mathcal{G}}}(k)} ||M_1 U^{-1} M_2^{-1}||\}. \qquad (4.81)$$

where $\theta = (a_{n-1} \cdots a_0 \; b_{n-1} \cdots a_0)^T$. However, such an approach yields two problems. First the optimization problem in (4.81) is rather complex. Then, even in the case where the optimization problem in (4.81) could be performed, it is very likely that the new inputs values at time $k+1$ will differ from the previous input values at time k. Hence it is far from clear how the $2n$-periodicity assumption of the input sequence can cope with such an approach.
For these reasons, we adopt a different strategy.
Suppose that (4.1) is excited by a $2n$-periodic input of the form (4.73) where $u_0, \cdots, u_{2n-1}$ satisfy (4.42) with $\gamma > 0$. Intuitively, if γ in (4.73) increases, then the output signal to output error (y_δ) ratio is improved and therefore the effect of modeling error on these results is reduced. Indeed, we have that $y(k) = \tilde{y}(k) + y_\delta(k)$, $\forall k$, where $\phi_\delta(k)$ is bounded according to (4.72), hence $|y(k)| \geq ||\tilde{y}(k)| - \overline{y}_\delta||$, $\forall k$. $||\tilde{\phi}(k)||$ can be made arbitrarily large by choosing γ sufficiently large, thus $|y(k)|$, and thus $||\phi(k)||$ can be made arbitrarily large. It follows from (4.7) that by choosing γ sufficiently large, the width of the hyperstrips $\mathcal{G}(k)$, $\forall k$, i.e., the dimensions of the uncertainty set can be made arbitrarily large. From this discussion, it follows that by choosing γ large enough, the radius of the uncertainty set can be made arbitrarily small. We formalize this result in the following theorem:

Theorem 4.3.12 (Radius of uncertainty set) *Suppose the input values $u_0, \cdots, u_{2n-1}$ to be such that* (4.42) *holds. Then, for any $\epsilon > 0$, and for any initial condition $\phi(0)$, there exists $\gamma_0 > 0$ such that for any value $\gamma \geq \gamma_0$, the radius of the smallest sphere containing $\hat{\mathcal{G}}(k)$ obtained when the system (4.1) is excited by the input (4.73) is smaller than ϵ, $\forall k \geq 2n-1$.*

Input sequence design for an arbitrarily small uncertainty set: algorithm

It follows from Theorem 4.3.12 that for any initial conditions $\phi(0)$, there exists a $2n$-periodic input of the form (4.73) such that the identified uncertainty set associated with the unknown system (4.1) is small enough so that the desired robustness criterion evoked in Section 4.2.2 is satisfied in finite time. However, since the real plant is unknown, only an iterative tuning of the parameter design γ in (4.73) can lead to this desired input. Unfortunately, if the design parameter γ is time-varying, the steady state input/output values also are time-varying, hence the periodicity of the input sequence is destroyed. For this reason, we increase the input gain slowly enough so that periodicity is approximated, but also such that it grows without bound. This leads to an input sequence of the form:

$$u_\gamma(k) = \gamma_k u_{t(k)}, \forall k, \tag{4.82}$$

where $u_0, \cdots, u_{2n-1}$ satisfy (4.42). The sequence $\{\gamma_k\}$ in (4.82) is strictly increasing and grows without bound, i.e.,

$$\gamma_k < \gamma_{k+1}, \forall k, \text{ and } \lim_{k\to\infty} \gamma_k = +\infty, \tag{4.83}$$

and it is asymptotically slow, i.e.,

$$\lim_{k\to\infty} (\gamma_{k+1} - \gamma_k) = 0. \tag{4.84}$$

We now introduce the main result of this chapter.

Theorem 4.3.13 (Input design) *Suppose the system to be of the form* (4.1) *where* $\theta^0 \in \mathcal{C}_n \cap \mathcal{S}_n$*. Define the input by* (4.82) *where the sequence* $\{\gamma_k\}$ *satisfies* (4.83) *and* (4.84) *and where* $u_0, \cdots, u_{2n-1}$ *satisfy* (4.42)*. Then, for any constant* $\epsilon > 0$*, there exists a time* $K \geq 2n-1$ *such that the uncertainty set* $\hat{\mathcal{G}}(k)$ *given in* (4.6) *is contained in a sphere with a radius smaller than* ϵ*,* $\forall k \geq K$*.*

Proof Suppose the input is of the form (4.82) and satisfies (4.42,4.83,4.84). Like in (4.67), we decompose the output signal $y(k)$ as follows:

$$y(k) = y_\gamma(k) + y_\delta(k), \tag{4.85}$$

where $y_\gamma u(k)$ is the output sequence we would obtain if the modeling error was zero ($\delta = 0$), and $y_\delta(k)$ is the output sequence given in (4.68), (4.70), we would obtain with the zero input ($u_\gamma = 0$). We now study the two terms y_γ and y_δ separately.

1. System without error modeling: we first consider the case where $\delta = 0$. Hence we have: $y_\delta = 0$. Let us consider any state space representation of (4.1) of the form:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu_\gamma(k) \\ y_\gamma(k) &= Cx(k), \end{aligned} \tag{4.86}$$

denoting by $x(k)$ the state vector at time k, and where A, B and C are with appropriate dimensions. We denote by $y_0, \cdots, y_{2n-1}$ the steady-state output values given in (4.26), corresponding to the $2n$-periodic input sequence $\{\tilde{u}(k), k = 0, \cdots, 2n-1\} = \{u_0, \cdots, u_{2n-1}\}$. Hence, $\tilde{x}(k), k = 0, \cdots, 2n-1$ is $2n$-periodic and describe the $2n$ possible states of the system (4.1) if the input is the $2n$-periodic input (4.14) with values $u_0, \cdots, u_{2n-1}$ and if the output takes the corresponding steady state values $\tilde{y}_0, \cdots, \tilde{y}_{2n-1}$. We have then for all k:

$$\begin{aligned} \tilde{x}(k+1) &= A\tilde{x}(k) + B\tilde{u}(k) \\ \tilde{y}(k) &= C\tilde{x}(k). \end{aligned} \tag{4.87}$$

Now, for all k, define the vector $\alpha_k \in \mathbb{R}^{2n-1}$ by

$$\alpha_k = x(k) - \gamma_k \tilde{x}(k). \tag{4.88}$$

Combining (4.82), (4.86) and (4.88), we get

$$\alpha_{k+1} = A\alpha_k + (\gamma_k - \gamma_{k+1})\tilde{x}(k+1). \tag{4.89}$$

Since the system (4.73) is asymptotically stable, A is strictly Schur stable. Moreover, $\tilde{x}$ is periodic, hence bounded. Therefore, it follows from (4.84) that:

$$\lim_{k\to\infty} \alpha_k = 0. \tag{4.90}$$

Thus, according to (4.88) we obtain that:

$$\lim_{k\to\infty} (x(k) - \gamma_k \tilde{x}(k)) = 0. \tag{4.91}$$

Therefore, since $\tilde{y}(k) = C\tilde{x}(k)$ and $y_\gamma(k) = Cx(k)$ we have:

$$\lim_{k\to\infty} (y_\gamma(k) - \gamma_k \tilde{y}(k)) = 0. \tag{4.92}$$

(4.92) means that asymptotically, the output sequence y_γ is equal to the output sequence $\tilde{y}$ we would obtain in steady-state if the input was $\tilde{u}$, multiplied by the gain sequence γ relating the actual input, u_γ to the $2n$-periodic sequence $\tilde{u}$. Since γ is chosen to be increasing asymptotically slow, (4.92) can be roughly interpreted as follows: the actual output sequence becomes arbitrarily close to a $2n$-periodic sequence with time. Roughly speaking, this intuitively means that our previous results, established in the case of a fixed $2n$-periodic input sequence with constant values can be applied asymptotically.

2. System with modeling error: we now suppose the modeling error δ to be non-zero. Combining (4.85) and (4.88), the regressor vector $\phi(k)$ defined in (4.2) can be rewritten as:

$$\phi(k) = \phi_\delta(k) + D_k\tilde{\phi}(k) + V_k, \forall k, \tag{4.93}$$

defining $\phi_\delta(k), \tilde{\phi}(k) \in \mathbb{R}^{2n}$, $D_k \in \mathbb{R}^{(2n)\times(2n)}$ and $V_k \in \mathbb{R}^{2n}$ by:

$$\phi_\delta(k) = (-y_\delta(k) \ \cdots \ -y_\delta(k-n+1)0\cdots0)^T, \ \forall k, \tag{4.94}$$

$$\tilde{\phi}(k) = (-\tilde{y}(k)\cdots - \tilde{y}(k-n+1) \ \tilde{u}(k)\cdots\tilde{u}(k-n+1))^T, \ \forall k, \tag{4.95}$$

$$D_k = \text{diag}(\gamma_k, \cdots, \gamma_{k-n+1}, \gamma_k, \cdots, \gamma_{k-n+1}), \tag{4.96}$$

$$V_k = (C\alpha_k, \cdots, C\alpha_{k-n+1}, 0\cdots, 0)^T. \tag{4.97}$$

Then, for any $k \geq 2n-1$, using (4.93) we can rewrite the matrix $R(k)$ in (4.9) as:

$$R(k) = \Delta(k) + \mathcal{V}_k + D_k Q_0 + E_k, \tag{4.98}$$

where Q_0 is given in (4.33), $\Delta(k)$ is given in (4.74) and for any $k \geq 2n-1$, the matrices $\mathcal{V}_k, E_k \in \mathbb{R}^{2n\times 2n}$ are given by:

$$\mathcal{V}_k = [V_k \cdots V_{k-1} \quad V_{k-2n+1}], \forall k \geq 2n-1,$$
$$E_k = [0\ (D_{k+1} - D_k)\tilde{\phi}(k+1)\ \cdots\ (D_{k-2n+1} - D_k)\tilde{\phi}(k-2n+1)]$$

where D_k is defined in (4.96). Now, it follows from (4.84) that

$$\lim_{k\to\infty} E_k = 0. \tag{4.99}$$

And using (4.90), we have that

$$\lim_{k\to\infty} \mathcal{V}_k = 0. \tag{4.100}$$

Moreover, by construction of the input (4.82), it follows from Theorem 4.3.5 that $\det(D_k Q_0) \neq 0$. Since the matrix $\Delta(k)$ is bounded in norm according to (4.74), by continuity of the determinant, we conclude from (4.98), (4.99) and (4.100) that there exists a time $K \geq 2n-1$ such that $\det(R(k)) \neq 0$, $\forall k \geq K$. From (4.8), this implies that the uncertainty set based on $2n$ successive measurements is bounded after a finite time. Finally, since the input gain grows without bound, $||\phi(k)||$ grows also without bound and therefore (4.7) implies that the radius of the uncertainty set based on $2n$ successive measurements can be made arbitrarily small. ■

Example 4.3.14 Examples of sequences $\{\gamma_k\}_{k\in\mathbb{N}}$ that satisfy (4.83) and (4.84) are $\gamma_k = \log k$ and $\gamma_k = \sqrt{k}$. We now illustrate our design method with an example in the first order case $n = 1$. We choose $(a_0, b_0) = (0.8, 3)$ to be the real unknown system, and the uncertainty δ is taken to be a random signal taking its values in $[-0.5, 0.5]$. We suppose that the known bounds on δ are such that $-\underline{\delta} = \overline{\delta} = 0.6$. Finally, we choose $u_0 = 1.5$ and $u_1 = 0.3$. For the gain sequence $\gamma_k = \sqrt{k}$, the set-membership identification using the input $u(k) = \gamma_k u_{t(k)}$, $t(k) = k \mod 2\ \forall k$ is illustrated in Figure 4.2. For the gain sequence $\gamma_k = \sqrt{k}$, the set-membership identification using the input $u(k) = \gamma_k u_{t(k)}$, $t(k) = k \mod 2\ \forall k$ is portrayed in Figure 4.3. In both cases, the obtained membership set is plotted for 30 iterations, together with a measure of the radius of the smallest outer bounding sphere centered in (a_0, b_0). As expected, both cases lead an uncertainty set which decreases uniformly with time. ■

Remark 4.3.15 Because of the increase of the input gain γ, our approach might not look appealing at first sight. However, since the system is unknown, no information tells us a priori what value this gain should take so that the uncertainty set becomes small enough to allow certainty equivalent control to be started. Hence to increase γ is somehow unavoidable. Having observed that increasing the gain is necessary for control purposes, it should also be emphasized that within a complete adaptive control scheme the input gain will never be increased more than necessary. Indeed, the idea is to no longer step γ_k as soon as the uncertainty set is small enough to be useful for control. It follows from Theorem 4.3.13 that this happens after a finite number of iterations, which guarantees that the input sequence stays bounded during identification.

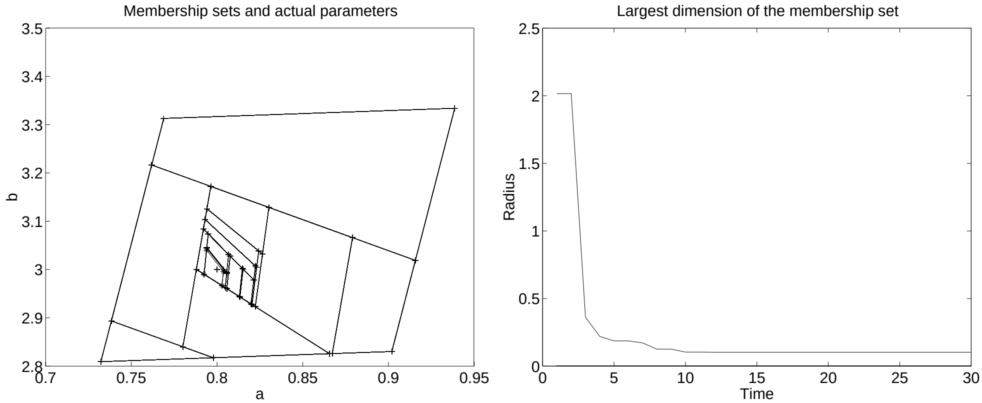


Figure 4.2: Set membership identification $\gamma_k = \sqrt{k}$, $(a_0, b_0) = (0.8, 3)$.

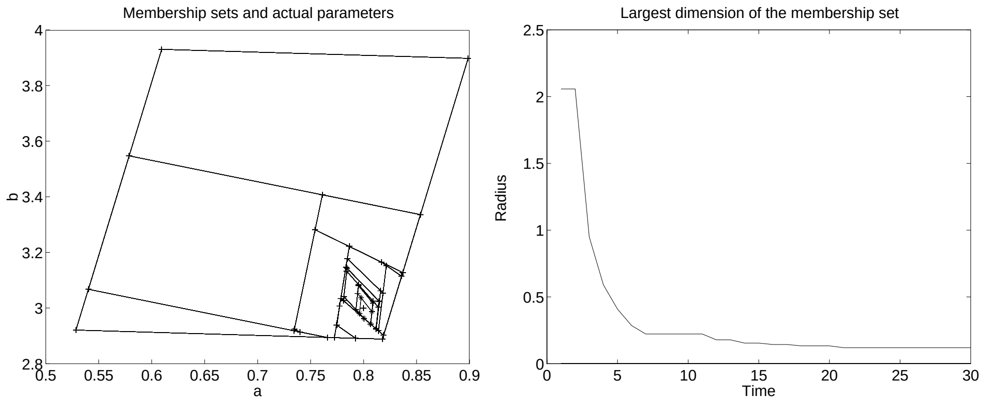


Figure 4.3: Set membership identification $\gamma_k =$log$k$, $(a_0, b_0) = (0.8, 3)$.

4.4 Conclusions

In this chapter an input sequence design has been proposed in the context of set-membership identification for control, to ensure that the uncertainty set becomes arbitrarily small with time. This guarantees that within finite time, a certainty equivalence type of control strategy can be safely started, 'safely' in the sense that the control design can rely on any model in the uncertainty set. Because the system is unknown, the time at which the designer can rely on the uncertainty set to control this unknown plant cannot be given a priori. However, the crucial point in our approach is that the input gain will not grow more than it is necessary for control. Other input structures may be better than $2n$-periodic inputs in the sense that a strongly robust uncertainty set would be obtained in less time or putting less energy in the identification input. However, the choice of a $2n$-periodic input structure is here motivated by its simple description and analysis.
In this approach, a unique design parameter γ is required, and the choice of this parameter is left free to the designer, provided that it increases without bound and is asymptotically slow. Therefore the computational complexity of the proposed strategy is rather low. Note however that the time at which the control phase can start depends explicitely on the increase rate of the design parameter and it is still not clear how to choose the sequence γ so that the uncertainty set shrinks fast enough to achieve strong robustness but at the same time slowly enough not to exceed the minimum input energy level required to achieve strong robustness.
In comparison with many input designs proposed in the literature ([1], [5], [11], [12]), our approach is not optimal in the sense that it is not based on the selection of inputs which would optimize some criterion related to the size of the uncertainty set. However, such optimal inputs would depend on the unknown system. Therefore these methods yield only estimations of such optimal inputs, and the "goodness" of these estimations is related to the uncertainty set. The larger the uncertainty set is, the poorer the estimate inputs are expected to be. For this reason, it is not clear whether they lead to a smaller uncertainty faster. This is an interesting problem which deserves further investigation. Finally, in this approach we considered the two following assumptions: the system to be controlled is asymptotically stable and its order n is exactly known. If any of these two assumptions is violated, then the guarantee that the uncertainty will shrink uniformly with time does not hold anymore. These are serious limitations which should be considered in future work.

Chapter 5

Strongly robust adaptive control

This chapter deals with adaptive control incorporating the notion of strong robustness introduced in Chapter 3. The process to be controlled belongs to the class of systems defined in Chapter 2 and the control objective belongs to the class of control objectives specified in Chapter 2. Adaptive control based on strong robustness splits in two phases: as long as a criterion checking strong robustness of the uncertainty set containing the true system to be controlled is not satisfied, no control action is undertaken and attention is paid to identification only. Once the above criterion is satisfied, then effort is put on control, by means of a classical certainty equivalence type of strategy. After presenting the general scheme of adaptive control based on strong robustness, analysis shows that the limitations exposed in Chapter 1 arising in standard certainty equivalence control systems are overcome by using such a control approach. In particular, when strong robustness is achieved, which is guaranteed to happen in finite time using the identification input designed in Chapter 4, no pole/zero cancellation phenomenon can occur. Moreover, the time-varying model-based controller will stabilize the true plant to be controlled, irrespectively of the adaptation speed.
Further, to shed some light on the introduced method, a more detailed analysis of strongly robust adaptive pole placement is given and a simulation example illustrates the effectiveness of this approach.

5.1 Introduction

In Chapter 1, we have seen that traditional adaptive control of linear time-invariant systems is based on the certainty equivalence principle. Rather simple from a computational point of view, this paradigm yet suffers from three main drawbacks. First controllability of the estimated is usually not guaranteed in practice, which may result in the paralysis of the adaptive control system. Secondly, insufficient initial knowledge on the true system might involve destabilizing model based controllers, in which case bad transients may be expected. Finally, the time variations of the model and hence of the model-based controller could be so fast as to disrupt asymptotic stability of the control system. To avoid these three undesirable phenomena, we ought to design an adaptive control scheme based on a test checking on-line whether we risk to meet these three problems exists or not so as to decide when to put more effort on

identification or control.
Suppose now that the following assumption holds.

Assumption 5.1.1 *The true system parameter θ^0 belongs to a known subset of systems in $\mathcal{G} \in \mathcal{P}_n$ and $\mathcal{G}$ is strongly robust with respect to the (given) control objective.*

Under Assumption 5.1.1, a traditional adaptive control scheme using certainty equivalence could be started, keeping in $\mathcal{G}$ the model on which to base the control design at any time. Then, due to strong robustness of the set $\mathcal{G}$ (see Section 2.2 in Chapter 2), the model sequence is guaranteed to keep controllability and to keep stabilizing the real unknown plant, no matter how fast adaptation might go. Unfortunately, Assumption 5.1.1 requires that a large information on the system is available, which is not satisfied most of the time in the initial set-up of adaptive control approaches. Hence, rather than supposing a priori that Assumption 5.1.1 holds, one can raise the following question: how to achieve the situation where we would know a strongly robust set of systems $\mathcal{P}_n$ containing the true system to be controlled? In this case, then the control task could be carried out using a traditional adaptive control strategy.

Inspired by this discussion, our aim is to design an adaptive control approach splitting in two phases [27]: as long as no conclusion can be drawn concerning strong robustness of the model set, which indicates that the danger of facing the above three undesirable effects exists, we do not undertake any control action. Instead, we collect information on the true system to be controlled, in such a way that strong robustness will be achieved in finite time (see Chapter 4). We shall call *identification phase* this first phase. Once a criterion indicating when the model set is strongly robust is satisfied, the adaptive system switches to *the control phase*. In this second phase, control can be started using a classical certainty equivalence strategy since from that time on, the model is and will stay controllable while the time-varying model-based controller is guaranteed to stabilize the true system at any time.

This chapter is organized as follows. We first formulate the problem statement this chapter deals with. Then, we describe the general scheme of adaptive control based on strong robustness. Further, the analysis of the algorithm is provided. Next we focus our interest on pole placement design based on strong robustness, for which a more detailed analysis and simulation examples are given. Finally, potential modifications of the general adaptive control scheme are addressed for further research.

5.2 Motivation

We first remind the assumptions made on the system to be controlled and on the control objective treated in this thesis (see Assumption 2.1.13).

Assumption 5.2.1 (System) *The system to be controlled is of the form*

$$y(k+1) = (\theta^0)^T \phi(k) + \delta(k), \phi(0),$$

where

$$\theta^0 = (a^0_{n-1} \ \cdots \ a^0_0 \, b^0_{n-1} \ \cdots \ b^0_0)^T \in \mathcal{C}_n \cap \mathcal{S}_n \tag{5.1}$$

is the unknown parameter vector and ϕ is the regressor vector given by

$$\phi(k) = (-y(k) \ \cdots \ -y(k-n+1) \ u(k) \ \cdots \ u(k-n+1))^T \in \mathbb{R}^{2n}. \tag{5.2}$$

The model uncertainty δ is unknown-but-bounded with known bounds $\underline{\delta}, \overline{\delta}$ (Assumption 2.1.10), i.e., is such that

$$\underline{\delta} \le \delta(k) \le \overline{\delta}, \forall k. \tag{5.3}$$

The control objective is fixed and satisfies the following assumption.

Assumption 5.2.2 (Controller) *The map*

$$f : \theta \in \mathcal{C}_n \longmapsto f(\theta) \in \mathbb{R}^{1\times(2n-1)} \tag{5.4}$$

assigning any system $\theta = (a_{n-1} \ \cdots \ a_0 \ b_{n-1} \ \cdots \ b_0)^T \in \mathcal{C}_n$ with its controller $f(\theta) \in \mathbb{R}^{1\times(2n-1)}$ leading to the control law

$$u(k) = f(\theta)x(k), \forall k, \tag{5.5}$$

where x is the state vector given by

$$x(k) = (y(k) \ \cdots \ y(k-n+1) \ u(k-1) \ \cdots \ u(k-n+1))^T \in \mathbb{R}^{2n-1}. \tag{5.6}$$

is single valued and continuous, and such that the resulting closed-loop system defined by

$$\begin{aligned} x(k+1) &= (A(\theta) + B(\theta)f(\theta))x(k) \\ y(k) &= Cx(k) \end{aligned} \tag{5.7}$$

is asymptotically stable (see Definition 3.1.1). We recall that $A(\theta)$, $B(\theta)$ and C are given by:

$$A(\theta) = \begin{bmatrix} -a_{n-1} & \cdots & \cdots & -a_1 & -a_0 & b_{n-2} & \cdots & \cdots & b_1 & b_0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & & \vdots & \vdots & \vdots & & & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots & \vdots & & & \vdots & \vdots \\ \vdots & & & 1 & \vdots & \vdots & & & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & & & 0 & \vdots & 1 & & & \vdots & \vdots \\ \vdots & & & \vdots & \vdots & 0 & \ddots & & \vdots & \vdots \\ \vdots & & & \vdots & \vdots & \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 1 & 0 \end{bmatrix} \tag{5.8}$$

$$B(\theta) = \begin{bmatrix} b_{n-1} & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}^T \tag{5.9}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^T. \tag{5.10}$$

We now formulate the problem addressed by this chapter.

Problem Statement 5.2.3 (Adaptive control objective) *The desired control objective is fixed and satisfies Assumption 5.2.2. Given the measurements* $\{u(k), y(k),\ k = 0, 1, 2, \cdots\}$ *generated by* (5.1)*, the adaptive control objective is twofold:*

- *generate a sequence of inputs such that asymptotically the applied inputs equal the inputs that would have been calculated on the basis of the true system parameters, i.e.,* $u(k) \to f(\theta^0)x(k)$ *as* $k \to \infty$ *with* f *defined in Assumption 5.2.2;*
- *do not allow the adaptive system to involve any destabilizing controller, at any time of the design.*

Remark 5.2.4 The controllability assumption in (5.1) is motivated by the control objective in Problem Statement 5.2.3. The assumption that the true system to be controlled (5.1) is open-loop asymptotically stable is for the sake of open-loop identification as we will see in the next section. Hence, as it follows from Assumption 5.2.2, the control objective is not only to keep the real plant stable but also to improve its performance. The assumption that the uncertainty δ is unknown-but-bounded according to (5.3) is chosen as the simplest case of approximate modeling. In particular, no stochastic assumptions on the modeling error are made. Moreover, the assumed structured of the uncertainty allows us to use the well-studied set-membership identification approach introduced in Section 2.2.

5.3 Strongly robust adaptive control: description

The general scheme of adaptive control systems based on strong robustness is depicted in Figure 5.1. As previously mentioned, two phases are distinguished in this control system scheme: the identification phase and the control phase. In the identification phase, data measurements are used to compute the set of all model candidate models that are consistent with these measurements. Then it is checked whether all the elements in this set are controllable or not. When controllability is guaranteed over the model set, it is then checked whether this set is strongly robust or not. When strong robustness is achieved, the system switches to the control phase where a classical certainty equivalence type of control strategy is applied: at each new measurement the model is updated within the strongly robust model set and the controller is designed on this model. Applying this controller on the actual plant leads to new input-output measurements, and subsequently, the uncertainty set, the model and the controller can be tuned more accurately so as to improve the closed-loop performance. Next, we describe in more details each block appearing in Figure 5.1.

5.3.1 The identification phase

During the identification phase, three tasks are completed at each new input/output measurement: the membership set is first computed, it is then checked whether all its elements are controllable or not. When controllability is achieved, then it is checked whether the membership set is strongly robust or not. Finally, if strong robustness is obtained over the whole model set, the adaptive scheme exits the first phase, otherwise new measurement data are collected and the three previous steps are re-iterated.

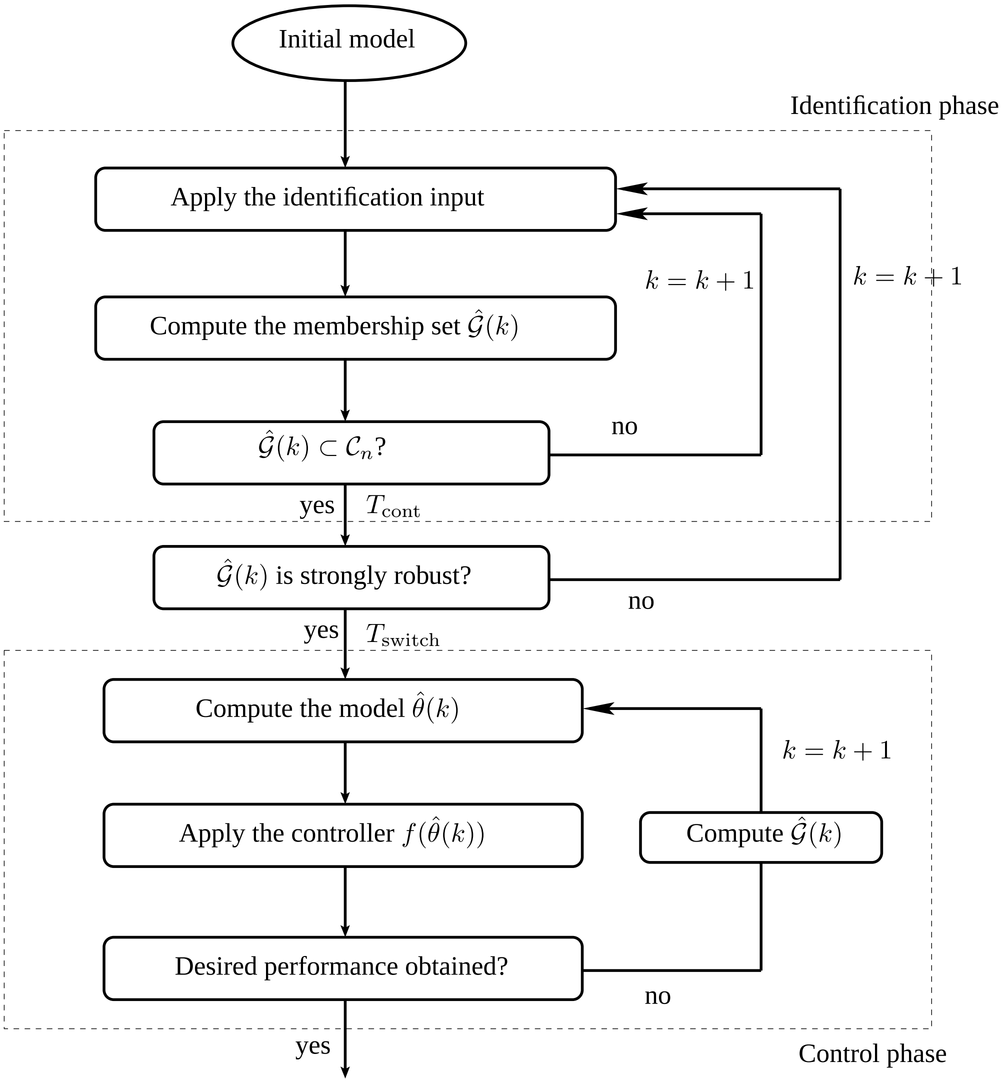


Figure 5.1: Strongly robust adaptive control: iterative scheme

Identification input

During the identification phase, we apply the $2n$-periodic input sequence constructed in Section 4.3.3, Chapter 4 defined as follows:

$$u_\gamma(k) = \gamma_k u_{t(k)}, \forall k, \tag{5.11}$$

where $u_0, \cdots, u_{2n-1}$ are arbitrarily chosen but satisfy the condition:

$$\gcd(\sum_{i=0}^{2n-1} u_i \xi, \xi^{2n} - 1) = 1, \tag{5.12}$$

and the gain sequence $\{\gamma_k\}$ in (5.11) is strictly increasing and grows without bound, i.e.,

$$\gamma_k < \gamma_{k+1}, \forall k, \text{ and } \lim_{k\to\infty} \gamma_k = +\infty, \tag{5.13}$$

and it is asymptotically slow, i.e.,

$$\lim_{k\to\infty} (\gamma_{k+1} - \gamma_k) = 0. \tag{5.14}$$

We motivate the choice of such an input in Section 5.4.1.

Compute the membership set $\hat{\mathcal{G}}(k)$

The membership set at time k contains all candidate models, described by all the parameter vectors θ consistent with the input/output data measurements $\{u(i), y(i)\}_{i\leq k}$ generated by the true system (5.1) when excited by the input (5.11, 5.13, 5.14). As shown in Section 2.2 in Chapter 2, $\hat{\mathcal{G}}(k)$ is the polyhedron defined by

$$\hat{\mathcal{G}}(k) = \bigcap_{i=1}^{k} \mathcal{G}(i), \tag{5.15}$$

where $\mathcal{G}(i)$, $i = 1, \cdots, k$ is defined by

$$\mathcal{G}(k) = \{\theta \in \mathcal{P}_n : \underline{\delta} \leq y(k) - \theta^T \phi(k-1) \leq \overline{\delta}\}. \tag{5.16}$$

Hence $\hat{\mathcal{G}}(k)$ is computed as the intersection of the $2k$ half-spaces in $\mathbb{R}^{2n}$ defined by:

$$\{\theta \in \mathcal{P}_n : y(k) - \theta^T \phi(k-1) \leq \overline{\delta}\}, \ \forall i \leq k \tag{5.17}$$

$$\{\theta \in \mathcal{P}_n : y(k) - \theta^T \phi(k-1) \geq \underline{\delta}\}, \ \forall i \leq k. \tag{5.18}$$

Remark 5.3.1 The set defined in (5.15) might contain uncontrollable systems or systems that are not asymptotically stable. However, from our prior knowledge, the true system θ_0 is open-loop asymptotically stable and controllable. Hence it seems that a ”good” model should also be open-loop asymptotically stable and controllable, in which case we could substitute to the model set $\hat{\theta}(k)$ obtained in (5.15) with its subset of controllable and asymptotically stable elements $\hat{\mathcal{G}}(k) \cap \mathcal{S}_n \cap \mathcal{C}_n$. This suggests to perform some projections of the model set $\hat{\mathcal{G}}(k)$ on the set $\mathcal{S}_n \cap \mathcal{C}_n$ so as to capture the features of the true system in the model set. Nevertheless, the reason why we do not proceed in this way is that the set $\mathcal{S}_n \cap \mathcal{C}_n$ is in general neither convex nor closed, hence the resulting intersection $\hat{\mathcal{G}}(k) \cap \mathcal{S}_n \cap \mathcal{C}_n$ might be in turn not convex nor closed. However, convexity of the model set are fundamental properties when performing a convex optimization method for the model estimation.

Check controllability of any model in $\hat{\mathcal{G}}(k)$

In contrast to a large number of adaptive control approaches [32], [33], [96], [70], the uncertainty set (5.15) is a priori not assumed to be a subset of the class of controllable systems. However, as we see further, the next step in the algorithm depicted in Figure 5.1, consisting in checking the strong robustness of the uncertainty set, requires that all models are controllable. As a consequence, it is fundamental that the set of estimate models $\hat{\mathcal{G}}(k)$, on which the control design will be based, is a subset of $\mathcal{C}_n$ in finite time.
We now proceed in two steps. We first show that the use of the identification (5.11, 5.13, 5.14) leads in finite time to an uncertainty set which is a subset of the set of controllable systems. Then, we derive a test that explicitely measures a time at which boundedness is achieved. Subsequent to this discussion, the algorithm which allows us to test controllability over the uncertainty set is given.

The first question we need to investigate is hence the following: how to choose an input sequence leading to a membership set whose elements are all controllable in finite time? Since the true system θ^0 is controllable, the distance from θ^0 to the set of uncontrollable systems in $\mathcal{P}_n$ is strictly positive, hence there exists an open neighborhood of θ^0 which is a subset of $\mathcal{C}_n$. Therefore, if the uncertainty set $\hat{\mathcal{G}}(k)$ is sufficiently small, i.e., if $\hat{\mathcal{G}}(k)$ is bounded in $\mathbb{R}^{2n}$ and if the radius of the smallest sphere containing $\hat{\mathcal{G}}(k)$ is sufficiently small, then $\hat{\mathcal{G}}(k) \subset \mathcal{C}_n$. Based on this discussion, we have the following result.

Theorem 5.3.2 (Identification input) *Given any set $\Omega \subset \mathcal{P}_n$, let $\rho(\Omega)$ denotes the radius of the smallest sphere containing Ω contained in $\mathcal{P}_n$. By convention, if Ω is not bounded, $\rho(\Omega) = \infty$. Consider the system given by* (5.1)*. Suppose that the identification input u is such that the membership set given in* (5.15) *satisfies:*

$$\lim_{k\to\infty} \rho(\hat{\mathcal{G}}(k)) = 0. \tag{5.19}$$

Then there exists T_1 such that $\hat{\mathcal{G}}(k) \subset \mathcal{C}_n$, $\forall k \geq T_1$.

Hence, it follows from Theorem 5.3.2 that when resorting to an input sequence satisfying Theorem 4.3.13 in Chapter 4, there exists a finite time T_1 such that $\hat{\mathcal{G}}(k) \subset \mathcal{C}_n$, $\forall k \geq T_1$.

After we have shown that all models in the uncertainty set become controllable in finite time if the input sequence is appropriately chosen, we now focus on the following problem: at each time instant k, how to check in practice whether or not $\hat{\mathcal{G}}(k) \subset \mathcal{C}_n$? Indeed, the time at which the condition $\hat{\mathcal{G}}(k) \subset \mathcal{C}_n$ holds should be computed, so that the next task of the algorithm starts in finite time. In this respect, the following result has been established in Chapter 3.

Theorem 5.3.3 *There exists a time $T_{\text{bound}} \geq 2n-1$ such that*

$$\det[\phi(T_{\text{bound}})\ \phi(T_{\text{bound}}-1)\cdots\phi(T_{\text{bound}}-2n+1)] \neq 0. \tag{5.20}$$

Proof: the proof directly follows from Theorem 4.3.13. ∎

We saw in Chapter 4 that (5.20) is equivalent to say that the intersection formed by the $2n$ hyperstrips $\mathcal{G}(T_{\rm bound}), \cdots, \mathcal{G}(T_{\rm bound} - 2n + 1)$ is bounded. Hence, (5.20) guarantees that the model set becomes bounded in finite time, and at the same time provides us with a way to check practically whether the uncertainty set is bounded or not. Moreover, we have: $\forall k \geq T_{\rm bound}$, $\hat{\mathcal{G}}(k)$ is bounded.

Now, once the test (5.20) allowing us to verify boundedness of the uncertainty set is fulfilled, we proceed as follows in order to check controllability of the members in the model set.

Algorithm 5.3.4 (Check controllability of any system in $\hat{\mathcal{G}}(k)$) *$\forall k \geq T_{\rm bound}$, denote by $\theta^*(k)$ the center of the smallest sphere of systems in $\mathcal{P}_n$ containing $\hat{\mathcal{G}}(k)$. Form the smallest orthotopic set $\hat{\mathcal{G}}_{\rm b}(k)$ of systems containing $\hat{\mathcal{G}}(k)$, with center $\theta^*(k)$. Recalling that any system $\theta \in \mathcal{P}_n$ has coordinates:*

$$\theta = (a_{n-1} \ \cdots \ a_0 \ b_{n-1} \ \cdots \ b_0)^T, \tag{5.21}$$

then $\hat{\mathcal{G}}_{\rm b}(k)$ is defined by

$$\begin{aligned}\hat{\mathcal{G}}_{\rm b}(k) = \{\theta \in \mathcal{P}_n : \ & a_i(k)^* - \Delta a_i(k) \leq a_i \leq a_i(k)^* + \Delta a_i(k), \\ & b_i(k)^* - \Delta b_i(k) \leq b_i \leq b_i(k)^* + \Delta b_i(k), i = 0, \cdots, 2n-1\},\end{aligned} \tag{5.22}$$

where the $2n$ dimensions $\Delta a_i(k) > 0, \Delta b_i(k) > 0$, $i = 0, \cdots, 2n-1$ of $\hat{\mathcal{G}}_{\rm b}(k)$ are such that

$$\Delta a_i(k) = \min\{\Delta > 0 : a_i(k)^* - \Delta \leq a_i \leq a_i(k)^* + \Delta, \forall \theta \in \hat{\mathcal{G}}(k)\}, \forall k \geq T_{\rm bound}, \tag{5.23}$$

and

$$\Delta b_i(k) = \min\{\Delta > 0 : b_i(k)^* - \Delta \leq b_i \leq b_i(k)^* + \Delta, \forall \theta \in \hat{\mathcal{G}}(k)\}, \forall k \geq T_{\rm bound}. \tag{5.24}$$

Further, apply Theorem 3.3.5 to check whether $\hat{\mathcal{G}}(k) \subset \mathcal{C}_n$ or not: form the Sylvester matrix $S_\Delta(k) \in \mathbb{R}^{(2n-1)\times(2n-1)}$ defined by

$$S_\Delta(k) = \begin{bmatrix} \Delta a_0(k) & \Delta a_1(k) & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \Delta a_0(k) & \cdots & \Delta a_{n-1}(k) & 1 & \ddots & \vdots \\ \vdots & & \ddots & & & \ddots & 0 \\ 0 & \cdots & 0 & \Delta a_0(k) & \Delta a_1(k) & \cdots & 1 \\ \vdots & & & \Delta b_0(k) & \Delta b_1(k) & \cdots & \Delta b_{n-1}(k) \\ \vdots & & \Delta b_0(k) & \Delta b_1(k) & \cdots & \Delta b_{n-1}(k) & 0 \\ 0 & \iddots & \iddots & & \iddots & \ddots & \\ \Delta b_0(k) & \Delta b_1(k) & \cdots & \Delta b_{n-1}(k) & 0 & \cdots & 0 \end{bmatrix}.$$

Similarly, form the Sylvester matrix $S^*(k)$ associated to the center system $\theta^*(k)$ defined by (5.25) *replacing the terms $\Delta a_i(k)$, $\Delta b_i(k)$ by $a_i^*(k)$ and $b_i^*(k)$ respectively. Denoting by $\underline{\sigma}$, $\overline{\sigma}$ the smallest and largest singular value respectively, we have the following result (see Theorem 3.3.5 in Chapter 3). If*

$$\underline{\sigma}(S_\Delta(k)) < \overline{\sigma}(S^*(k)), \tag{5.25}$$

then $\hat{\mathcal{G}}(k) \subset \mathcal{C}_n$.

Now, it follows from Theorem 5.3.2 and Theorem 4.3.13 that when resorting to the input sequence (5.11, 5.13, 5.14), there exists a finite time $T_{\rm cont}$ such that (5.25) is satisfied $\forall k \geq T_{\rm cont}$. Hence $\hat{\mathcal{G}}(k) \subset \mathcal{C}_n$, $\forall k \geq T_{\rm cont}$.

Remark 5.3.5 The test proposed in Algorithm 5.3.4 to check whether a given uncertainty set is in the set of controllable systems or not might be quite conservative, firstly because we enclose the set to be tested in an outer bounding set, but also because (5.25) only provides us with a sufficient condition for non-singularity of the Sylvester matrices associated with any model in the uncertainty set. Therefore, if the initial set of systems contains nearly uncontrollable systems, it might happen that the set to be tested is contained in the set of controllable systems whereas the outer-bounding polytope (5.22) contains uncontrollable systems. However, the input (5.11, 5.13, 5.14) guarantees that the uncertainty set, and therefore the outer-bounding polytopic set (5.22), become sufficiently small in finite time. Hence using such an input ensures that (5.25) holds in finite time.

Achieve and check strong robustness

The test indicating when to switch to the second phase of the algorithm consists in checking whether $\hat{\mathcal{G}}(k)$ is strongly robust or not. Clearly, since the objective is to perform control of the unknown plant, we must guarantee that the control phase starts in finite time, i.e., we must ensure that the model set $\hat{\mathcal{G}}(k)$ is strongly robust in finite time. This remark raises two key issues: first, it is crucial that the identification input yields a strongly robust model set in finite time. Then, we must be able to test practically at each measurement whether the model set is strongly robust or not. We now discuss these two issues and subsequently derive an algorithm to test strong robustness of the uncertainty set.
Let us first concentrate on the following question: how to choose an input sequence leading to a strongly robust membership set in finite time? Here we recall the following theorem established in Chapter 3.

Theorem 5.3.6 (Existence of strongly robust open sets of systems) *Around any system $\theta^0 \in \mathcal{C}_n$ there exists an open strongly robust neighborhood of systems in $\mathcal{C}_n$.*

In the sequel, if the uncertainty set $\hat{\mathcal{G}}(k)$ is sufficiently small, i.e., if $\hat{\mathcal{G}}(k)$ is bounded in $\mathbb{R}^{2n}$ and if the radius of the smallest sphere containing $\hat{\mathcal{G}}(k)$ is sufficiently small, then it is strongly robust. Based on this discussion, we have the following result.

Theorem 5.3.7 (Identification input) *Given any set $\Omega \subset \mathcal{P}_n$, let $\rho(\Omega)$ denotes the radius of the smallest sphere containing Ω contained in $\mathcal{P}_n$. By convention, if Ω is not bounded, $\rho(\Omega) = \infty$. Consider the system given by* (5.1)*. Suppose that the identification input u is such that the membership set given in* (5.15) *satisfies:*

$$\lim_{k \to \infty} \rho(\hat{\mathcal{G}}(k)) = 0. \tag{5.26}$$

Then there exists $T_{\rm SR}$ such that $\hat{\mathcal{G}}(k)$ is strongly robust, $\forall k \geq T_{\rm SR}$.

Therefore, in the case where the input sequence is such that $\hat{\mathcal{G}}(k)$ is bounded and shrinks uniformly with time, a strongly robust uncertainty set is identified in finite time, hence the

adaptive system described in Figure 5.1 switches to the control phase in finite time. As matter of fact, it has been shown in Section 4.3, Chapter 4 that the input (5.11, 5.13, 5.14) is such that $\hat{\mathcal{G}}(k)$ is bounded and shrinks uniformly with time.
It is important to point out, and even emphasize, that after the adaptive scheme has switched to the control phase, identification proceeds in a passive way, it is subject to control. Hence, the identification input involved in Theorem 5.3.7 is truncated in the finite switching time, meaning that we do not require the uncertainty set to shrink indefinitely, i.e., we do not require the the unknown parameter vector θ^0 to be identified exactly.

Not only it is crucial to ensure that the membership set is strongly robust in finite time, but the adaptive scheme should also be able to measure at what time this condition is satisfied, so that the switch can be activated. Otherwise we loose all the benefit that the introduction of the notion of strong robustness in adaptive control may bring. Hence, it is fundamental to have an explicit test to check at each measurement whether the updated unfalsified set is strongly robust or not. We remind that at this step of the design, any model in the membership set is guaranteed to be controllable. Hence, the desired test amounts at checking a each time if a given set of controllable systems is strongly robust. The construction of a necessary and sufficient test for characterizing strongly robust sets in $\mathcal{C}_n$ is not trivial, and still requires further investigation. However, note that what we essentially need is a sufficient test for strong robustness. In this respect, we recall the following theorem obtained in Chapter 3.

Theorem 5.3.8 *The set $\hat{\mathcal{G}}(k) \subset \mathcal{C}_n$ is strongly robust if the following inequality holds:*

$$\forall \theta_1, \theta_2 \in \hat{\mathcal{G}}(k),\ \|f(\theta_2) - f(\theta_1)\| \leq r_{\mathbb{C}}(A(\theta_1) + B(\theta_1)f(\theta_1), B(\theta_1, I_{2n-1}), \qquad (5.27)$$

where $\forall \theta \in \mathcal{C}_n$, the matrices $A(\theta)$ and $B(\theta)$ are given in (5.8), (5.9) *and $f(\theta)$ is the controller given in Assumption 5.2.2 and $r_{\mathbb{C}}(A(\theta)+B(\theta)f(\theta), B(\theta, I_{2n-1})$ denotes the complex stability radius of the matrix $(A(\theta) + B(\theta)f(\theta))$ with respect to the perturbation structure $(B(\theta), I_{2n-1})$ (see Definition 3.2.3).*

We have the following result.

Theorem 5.3.9 *If the identification input sequence $\{u(k)\}$ is such that $\lim_{k\to\infty} \rho(\hat{\mathcal{G}}(k)) = 0$, where $\rho(\hat{\mathcal{G}}(k))$ denotes the radius of the smallest sphere containing $\hat{\mathcal{G}}(k)$, then there exists k_1 such that* (5.27) *is satisfied.*

Proof: suppose the identification input to be such that $\lim_{k\to\infty} \rho(\hat{\mathcal{G}}(k)) = 0$. It follows from Remark 5.3.5 that there exists a time K such that $\forall k \geq K$, $\hat{\mathcal{G}}(k) \subset \mathcal{C}_n$. Then by continuity of the map f, we have that $\lim_{k\to\infty} \rho(f(\hat{\mathcal{G}}(k))) = 0$, where $\rho(f(\hat{\mathcal{G}}(k)))$ denotes the radius of the smallest sphere containing $f(\hat{\mathcal{G}}(k))$ for $k \geq K$. Therefore, $\forall \epsilon > 0$, $\exists k_1 \geq K$ such that $\forall f, f' \in \hat{\mathcal{G}}(k_1)$ then $||f - f'|| \leq \epsilon$. Now, choose any $\theta_1 \in \hat{\mathcal{G}}(k_1)$ and $\epsilon = r^{\mathbb{C}}_{A_1+B_1 f(\theta_1)}$, where $r^{\mathbb{C}}_{A_1+B_1 f(\theta_1)}$ denotes the complex stability radius of the Schur matrix $A_1 + B_1 f(\theta_1)$ with respect to the perturbation structure (B_1, I_{2n-1}) as defined in Definition 3.2.3 where A_1 and B_1 are obtained on the basis of θ_1 according to (5.8) and (5.9)respectively. We then have: $\forall f, f' \in \hat{\mathcal{G}}(k_1)$ then $||f - f'|| \leq r^{\mathbb{C}}_{A_1+B_1 f(\theta_1)}$. Hence, $\forall \theta_2 \in \hat{\mathcal{G}}(k_1)$ we have: $||f(\theta_1) - f(\theta_2)|| \leq r^{\mathbb{C}}_{A_1+B_1 f(\theta_1)}$, this for any $\theta_1 \in \hat{\mathcal{G}}(k_1)$. Thus (5.27) is satisfied in $k = k_1$. Clearly, this implies that (5.27) is satisfied, $\forall k \geq k_1$. ■

The preceding discussion yields the following algorithm.

Algorithm 5.3.10 (Check strong robustness of $\hat{\mathcal{G}}(k)$) *$\forall k \geq T_{\text{cont}}$, if*

$$\forall \theta_1, \theta_2 \in \hat{\mathcal{G}}(k),\ \|f(\theta_2) - f(\theta_1)\| \leq r_{\mathbb{C}}(A(\theta_1) + B(\theta_1)f(\theta_1), B(\theta_1, I_{2n-1}), \tag{5.28}$$

using the notation of Theorem 5.3.8, then $\hat{\mathcal{G}}(k)$ is strongly robust.

The identification input design (5.11, 5.13, 5.14) provides us with an uncertainty set that satisfies Theorem 5.3.9. Hence the sufficient test for strong robustness (5.28) is satisfied in finite time, allowing the overall scheme to switch to the control phase in finite time. Let T_{switch} denote this switching time.

Remark 5.3.11 We clearly have: $2n \leq T_{\text{bound}} \leq T_{\text{cont}} \leq T_{\text{SR}} \leq T_{\text{switch}}$.

5.3.2 The control phase

In this part we suppose that the set $\hat{\mathcal{G}}(k)$ defined in (5.15) has been shown to be strongly robust, i.e., $k \geq T_{\text{switch}}$. The control phase is then started and relies on a classical certainty equivalence type of strategy. During this phase, two main tasks are performed: the model is updated, and the controller is designed on the basis of this estimate. Applying this controller to the real system leads then to new data measurement on the basis of which the membership set will be updated.

Compute the model $\hat{\theta}(k)$: at each measurement, the model $\hat{\theta}(k)$ of the true parameter vector θ^0 is updated, leading to the new model $\hat{\theta}(k+1)$. Since $\theta_0 \in \hat{\mathcal{G}}(k)$, we naturally choose $\hat{\theta}(T_{\text{switch}})$ as a member of $\hat{\mathcal{G}}(T_{\text{switch}})$, and $\forall k \geq T_{\text{switch}}$, the model $\hat{\theta}(k+1)$ is computed as the orthogonal projection of the previous estimate $\hat{\theta}(k)$ on the set $\hat{\mathcal{G}}(k+1)$ presented in Section 2.2.3, Chapter 2. As discussed in Remark 5.3.1, the convexity of $\hat{\mathcal{G}}(k)$ is crucial here in order to use orthogonal projection. This leads to the following update procedure introduced in Chapter 3:

$$\begin{aligned} &\hat{\theta}(T_{\text{switch}}) \text{ is arbitrarily chosen in } \hat{\mathcal{G}}(T_{\text{switch}}); \\ &\hat{\theta}(k+1) = \arg \min_{\theta \in \hat{\mathcal{G}}(k+1)} \{(\theta - \hat{\theta}(k))^T(\theta - \hat{\theta}(k))\},\ \forall k \geq T_{\text{switch}}. \end{aligned} \tag{5.29}$$

Hence, this update law is such that if the new measurement at time k does not bring any new information pertaining the updating of the set $\hat{\mathcal{G}}(k)$, then the model $\hat{\theta}(k)$ is not updated.

Remark 5.3.12 If $||\phi(k-1)|| = 0$ in (5.16), then the set $\mathcal{G}(k)$ is empty. Hence, if the new model at time k was obtained as the projection of the previous model on the new set $\mathcal{G}(k)$ (as it is done, e.g., in [72]), the case where $||\phi(k-1)|| = 0$ would cause numerical problem. This problem does not occur in our approach since at each time in the control phase, the new estimate $\hat{\theta}(k)$ is obtained by orthogonal projection of the previous estimate on the non-empty set $\hat{\mathcal{G}}(k)$ given in (5.15). According to (5.29), if $||\phi(k-1)|| = 0$ occurs (which is a priori possible in the control phase), $\mathcal{G}(k) = \emptyset$, hence $\hat{\mathcal{G}}(k) = \hat{\mathcal{G}}(k-1) \neq \emptyset$ and hence $\hat{\theta}(k) = \hat{\theta}(k-1)$ is defined.

Design a controller: $\forall k \geq T_{\mathrm{switch}}$, we have: $\hat{\mathcal{G}}(k) \subset \mathcal{C}_n$, hence a controller can be based on the model at any time. Following Assumption 5.2.2, at each time $k \geq T_{\mathrm{switch}}$, we compute the controller $f(\hat{\theta}(k))$, leading to the updated control input law:

$$u(k) = f(\hat{\theta}(k))x(k), \forall k \geq T_{\mathrm{switch}} \tag{5.30}$$

This controller is then applied to the real system (5.1) and a new data measurement $(y(k+1), \phi(k))$ is obtained from the real closed-loop system:

$$\begin{aligned} y(k+1) &= (\theta^0)^T \phi(k) + \delta(k) \\ u(k) &= f(\hat{\theta}(k))x(k), \end{aligned} \tag{5.31}$$

where the regressor vector ϕ and the state vector x are given by

$$\phi(k) = (-y(k), \cdots, -y(k-n+1), u(k), \cdots, u(k-n+1))^T \in \mathbb{R}^{2n}, \tag{5.32}$$

and (5.6) respectively. Based on the newly measured data, the new membership-set is updated according to (5.15).

5.4 Strongly robust adaptive control: analysis

This section is devoted to the analysis of the adaptive control scheme proposed in Section 5.3.

5.4.1 Finite switching time

The identification input (5.11, 5.13, 5.14) is constructed so that the uncertainty set is proved to strongly robust in finite time (T_{switch}). Therefore the control phase is guaranteed to start in finite time. Obviously, this switching time depends on the characteristics of the system to be controlled (initial conditions and value of the unknown parameter vector θ^0), on the characteristics of the uncertainty δ and on the chosen identification input. Intuitively, the switch from the identification phase to the control phase is expected to occur faster (in time or in term of energy level of the identification input) in the case of an uncertainty which is small in norm compared with the measured data. In contrast, for large uncertainty level (and conservative bounds $\underline{\delta}, \overline{\delta}$) we expect that more measurements or a higher input energy level will be necessary before the system switches to the control phase. But it seems quite natural that little prior knowledge on the system to be controlled requires a longer learning phase or a higher cost in terms of energy put in the identification input.

5.4.2 Convergence of the model to the real system

The convergence of $\hat{\theta}(k)$ to the true parameter vector is not a-priori guaranteed and is dependent on the input-output data that specify the uncertainty set $\hat{\mathcal{G}}(k)$. Note that the proposed scheme has the property of *neutrality*, i.e., in the case where the present uncertainty set is not falsified by the new input-output measurement, the model and hence the model-based controller are not updated. Hence, the adaptation process might stop during the control phase,

leading to a frozen adaptive system. Of course, such a case does by no means imply that the model error is zero, but simply indicates that the newly observed data do not bring any useful information with respect to the identification process.
Now, we emphasize that the quality of the model is measured by its control performance and not by its closeness to the real system. Indeed, the distance between the true model and its estimate might be very small, while the true system might be controllable and the estimate not controllable. Hence, a model is good enough if it leads to good control performance. In the sequel, convergence of the model to the real system is neither guaranteed nor necessary in the presented adaptive control approach.
The model update law (5.29) provides the following properties [72]:

Property 5.4.1 *The model error sequence $\{\hat{\theta}(k) - \theta^0\}$ is bounded and non increasing:*

$$\| \hat{\theta}(k) - \theta^0 \| \leq \| \hat{\theta}(k-1) - \theta^0 \|, \ \forall k \tag{5.33}$$

and is asymptotically slow, i.e,

$$\lim_{k \to \infty} (\| \hat{\theta}(k) - \theta^0 \| - \| \hat{\theta}(k-1) - \theta^0 \|) = 0. \tag{5.34}$$

It hence follows from these two properties that the parameter vector converges:

$$\exists \bar{\theta} \in \mathcal{C}_n : \lim_{k \to \infty} \hat{\theta}(k) = \bar{\theta}. \tag{5.35}$$

However, $\bar{\theta} = \theta^0$ does not necessarily holds.

5.4.3 Transient analysis

The proposed adaptive control approach mainly differs from classical approaches in the first phase, therefore the transient analysis is key in its analysis. Intuitively, since at no time the adaptive control system based on strong robustness involves any destabilizing controller, the transient behavior is expected to be superior to classical certainty equivalence-based schemes where, in contrast, the model-based controller may be temporarily destabilizing. A rigorous proof of this intuitive result in the case of pole placement can be found in the next section of this chapter.
In addition it is worth recalling that, even in the case where at each frozen time instant the closed-loop system would be obtained, stability of the time-varying system is not necessarily maintained in classical approaches if adaptation is too fast. In comparison, at no time in our strategy a destabilizing controller is applied to the real system, and closed-loop stability is guaranteed, regardless how fast the adaptation goes. Of course, the use of the identification input of the type (5.11,5.13, 5.14) may still generate poor transients, but this appears to be the inevitable price to be paid due to insufficient prior knowledge, whereas in classical adaptive control approaches the bad transients serve no purpose.

5.4.4 Asymptotic analysis

Once the adaptive system has switched to the control phase, which is guaranteed to occur in finite time, a classical adaptive control approach is used. Hence the asymptotic analysis is fairly standard [72]. The main characteristic of the asymptotic behavior, in the case of adaptive Pole placement design [72] and adaptive Linear Quadratic control design [95], is that the applied control law converges to the control we would obtain when using the real system parameter. Furthermore, in contrast with classical adaptive control where it must be a priori assumed that for all k, the model $\hat{\theta}(k)$ is controllable, and that all the limit points of the model sequence $\{\hat{\theta}(k)\}$ are controllable, we do not have this limitation. Indeed in our design, as soon as the control phase starts, controllability of the model and of all the limit points of the model sequence are guaranteed. In addition, at no time of our design, the time-variations induced by the adaptation process cannot destroy stability of the closed-loop system. Hence, bad asymptotic behavior caused by fast adaptation cannot occur.

5.4.5 Bounded input

It is shown in Chapter 4 that for the sake of the identification of a strongly robust uncertainty set in the identification phase, the input energy level must be increased so that the criterion for strong robustness described in Chapter 3 is satisfied. By construction of the identification input (5.11,5.13, 5.14), strong robustness is satisfied in finite time, hence after such time the identification phase stops, i.e., this identification input does not necessarily have to be used any longer. This implies that the input sequence stays bounded in the first phase. In the second phase, the input is designed according to (5.30), which also stays bounded since the estimate on which the controller is designed is guaranteed to be controllable. Therefore, boundedness of the input sequence u is guaranteed in both phases of the design.

5.4.6 Control performance

Since control performance is our ultimate goal, we have to be able to measure the performance of the closed-loop adaptive system. When the performance of the system is considered good enough, then adaptation of the parameter might stop. In general, measuring the closed-loop performance of the system is not a trivial task. However, in the case of pole assignment, this can be easily done by measuring the actual (time-varying) closed loop poles of the actual system given in (5.31). When these poles are close enough to the desired ones, one might freeze the adaptation procedure.

5.5 Strongly robust adaptive pole placement

After having presented the general philosophy of our approach, we feel that the best way to give some insight in the introduced approach is to explicit the algorithm in a more specified case, and we choose one of the most popular adaptive control problem: pole assignment.

Problem Statement 5.5.1 (Adaptive pole assignment) *The system to be controlled is described by*

$$y(k+1) = (\theta^0)^T \phi(k) + \delta(k),\ \forall k, \tag{5.36}$$

where $\theta^0 \in \mathcal{C}_n \cap \mathcal{S}_n$ *is the unknown parameter vector of the form* (5.1), ϕ *is the regressor vector given by* (5.2) *and* $\delta(k)$ *is the uncertainty at time* k *satisfying* (5.3).
Given the measurements $\{u(k), y(k),\ k = 0, 1, 2, \cdots\}$, *generate a sequence of inputs such*

that asymptotically the applied inputs equal the inputs that would have been calculated on the basis of the true system parameters, i.e., $u(k) \to f(\theta^0)x(k)$ *as* $k \to \infty$ *where* $f(\theta^0)$ *is the unique controller such that the closed-loop poles of the system defined by*

$$y(k+1) = (\theta^0)^T \phi(k) \quad (5.37)$$
$$u(k) = f(\theta^0)x(k)$$

are located in the desired stable poles $\{\alpha_i\}_{i=1,\cdots,2n-1}$, $|\alpha_i| < 1$. *Moreover, no controller destabilizing the true system should be involved, at any time in the design.*

We now give the algorithm of adaptive pole placement based on strong robustness

Algorithm 5.5.2 (Adaptive pole placement based on strong robustness)

Initial conditions and fixed parameters
$\underline{\delta}$; $\overline{\delta}$; $\phi(0)$; $\{\alpha_i\}_{i=1,\cdots,2n-1}$; $\hat{\mathcal{G}}(0) = \mathbb{R}^{2n}$;

Design parameters
$\{\gamma_k\}_{k\in\mathbb{N}} : \gamma_k \uparrow^{+\infty}, \gamma_k > 0, \lim_{k\to\infty}(\gamma_{k+1} - \gamma_k) = 0$;
$(u_0, u_1, \cdots, u_{2n-1}) \in \mathbb{R}^{2n} : \gcd(\sum_{i=0}^{2n-1} u_i\xi^i, \xi^{2n} - 1) = 1.$

1. Identification phase, $\forall k \geq 0$

- *apply to the real system* (5.36) *the identification input designed in Chapter 4 given by*

$$u(k) = \gamma_k u_{t(k)}, \text{ with } t(k) = k \mod 2n, \ \forall k. \quad (5.38)$$

where $u_0, \cdots, u_{2n-1}$ *are arbitrarily chosen but satisfy the condition:*

$$\gcd(\sum_{i=0}^{2n-1} u_i\xi, \xi^{2n} - 1) = 1, \quad (5.39)$$

and the gain sequence $\{\gamma_k\}$ *in* (5.11) *is such that*

$$\gamma_k < \gamma_{k+1}, \forall k, \text{ and } \lim_{k\to\infty} \gamma_k = +\infty, \quad (5.40)$$

and

$$\lim_{k\to\infty} (\gamma_{k+1} - \gamma_k) = 0. \quad (5.41)$$

- *measure* $(y(k+1), \phi(k))$ *from* (5.36) *and update the membership set* $\hat{\mathcal{G}}(k)$ *into*

$$\hat{\mathcal{G}}(k+1) = \hat{\mathcal{G}}(k) \cap \{\theta : \underline{\delta} \leq y(k+1) - \theta^T\phi(k) \leq \overline{\delta}\}; \quad (5.42)$$

2. Check controllability

- *apply Theorem 3.3.5. If* (2.39) *is not satisfied, re-iterate* ***1.***, *otherwise* $T_{\text{cont}} = k+1$ *and go to* ***2.***.

3. Check strong robustness: $\forall k \geq T_{\text{cont}}$

- *apply Theorem 5.3.8. If* (5.27) *is not satisfied, then re-iterate **1.** and **3.**. If* (5.27) *is satisfied,* $T_{\text{switch}} = k$ *and go to **4.**.*

4. Control phase, $\forall k \geq T_{\text{switch}}$

- *compute the model:*

$$\hat{\theta}(T_{\text{switch}}) \text{ is arbitrarily chosen in } \hat{\mathcal{G}}(T_{\text{switch}});$$
$$\hat{\theta}(k+1) = \arg \min_{\theta \in \hat{\mathcal{G}}(k+1)} \{(\theta - \hat{\theta}(k))^T(\theta - \hat{\theta}(k))\},\ \forall k \geq T_{\text{switch}}. \tag{5.43}$$

- *apply the control input*

$$u(k) = f(\hat{\theta}(k))x(k) \tag{5.44}$$

on the real plant (5.36) *where* x *is given in* (5.6) *and*

$$f(\hat{\theta}(k)) = F(A(\hat{\theta}(k)), B(\hat{\theta}(k))) \tag{5.45}$$

where F *is computed according Ackermann's formula:*

$$F(A, B) = -[0\ \cdots\ 0\ 1][B\ AB\ \cdots\ A^{2n-2}B]\Pi(A), \tag{5.46}$$

where $\Pi(\xi) = \sum_{i=1}^{2n-1}(\xi - \alpha_i)$ *is the desired closed-loop polynomial, and* $A(\hat{\theta}(k))$, $B(\hat{\theta}(k))$ *are given in* (2.9), (2.10) *replacing* θ *by* $\hat{\theta}(k)$.

- *measure* $(y(k+1), \phi(k))$ *and compute the model set* $\hat{\mathcal{G}}(k)$ *according to* (5.42).
- $k \to k+1$ *and re-iterate **4.** until the closed performance of the system formed by* (5.36), (5.44) *is satisfactory.*

5.5.1 Asymptotics

The asymptotic analysis elements given in Subsection 5.4.4 apply here. The main (and desired) feature of the controlled behavior is that asymptotically, the applied control input equals the desired control input we would obtain on the basis of the real system parameters.

$$\lim_{k \to \infty} || \frac{u(k) - f(\theta^0)x(k)}{||x(k)||} || = 0, \tag{5.47}$$

where u is computed according to (5.44), (5.45) [72].
Moreover, as soon as the control phase **4.** starts, controllability of the model (5.43) at any time and of all the limit points of the generated model sequence are guaranteed. In addition, time-variations induced by adaptation do not endanger stability of the closed-loop system (5.36, 5.44). Hence, bad asymptotic behavior caused by fast adaptation cannot occur.

5.5.2 Transient analysis

Our aim is now to show that the adaptive system described in Algorithm 5.5.2 has a better transient behavior than the classical certainty equivalence-based pole placement ([72], Chapter 4) when both methods are performed on the same initial system (5.1). For clarity of presentation, we first treat the first order case, i.e., $n = 1$. Let us first briefly describe the classical approach we are going to study.

Classical adaptive pole placement in the first order case

The system to be controlled is supposed to be given by

$$y(k+1) + a^0 y(k) = b^0 u(k), \forall k \tag{5.48}$$

where $\theta^0 = (a^0, b^0)$ is the unknown parameter vector in $\mathcal{S}_1 \cap \mathcal{C}_1$, i.e., such that $|a^0| < 1$ and $b^0 \neq 0$. Hence, the description (5.48) is a special case of the general case described by (5.36), assuming the uncertainty δ to be zero.

The adaptive control objective is to design an input sequence $u(k)$ that asymptotically converges to the input sequence $u^0(k)$ we would obtain on the basis of the true parameter values:

$$u^0(k) = \frac{\alpha + a^0}{b^0} y(k), \tag{5.49}$$

where α,$|\alpha| < 1$ is the given desired closed-loop pole.

The classical adaptive scheme proposed in [72] is the following.

Algorithm 5.5.3 (Classical adaptive pole assignment in the first order case)

Initial conditions $\hat{\theta}(0) = (\hat{a}(0), \hat{b}(0)) \in \mathbb{R}^2$, $y(0)$.

Recursion $\forall k \geq 0$

- *apply the control input given by*

$$u(k) = \frac{\alpha + \hat{a}(k)}{\hat{b}(k)} y(k) \tag{5.50}$$

  where $\hat{b}(k) \neq 0$ *is assumed to be satisfied thoughout the recursion.*

- *measure* $y(k+1)$ *and compute*

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{\phi(k)}{||\phi(k)||^2}(y(k+1) - [\hat{\theta}(k)]^T \phi(k)), \tag{5.51}$$

  taking $\phi(k) = (-y(k), u(k))$

- $k \rightarrow k+1$ *until the closed-loop performance of the system consisting of* (5.48), (5.50) *is satisfactory.*

Transient behaviors: a comparison

Clearly, the first problem when performing the adaptive scheme in Algorithm 5.5.3 is that controllability of the estimate is not always guaranteed, as nothing prevents $\hat{b}(k)$ to be zero in some times. Hence, without appropriate precautions, the adaptive scheme may be completely paralysed. Various modifications of Algorithm 5.5.3 have been proposed in order to solve this problem. For instance, we refer to [91] where the following approach is adopted: at each iteration one compare the absolute value of the coefficient $\hat{b}(k)$ to a fixed positive value ϵ_k, where the sequence $\{\epsilon_k\}_{k\in\mathbb{N}}$ is a priori given and strictly positive and decreases with time. If $\hat{b}(k) \leq \epsilon_k$, then one proceeds as in Algorithm 5.5.3. If $\hat{b}(k) < \epsilon_k$, then a modified input is applied so as to drive $\hat{b}(k+1)$ away from the critical value 0.

Now, even in the case where the condition $\hat{b}(k) \neq 0$ is always guaranteed, Algorithm 5.5.3 might lead to arbitrarily bad undesired transients. In this respect we have the following result.

Proposition 5.5.4 (Arbitrarily destabilizing controller) *For any system* $(a^0, b^0) \in \mathcal{S}_1 \cap \mathcal{C}_1$ *described by* (5.48)*, for any initial condition* $y(0)$*, for any desired pole location* α*,* $|\alpha| < 1$*, and for any integers* $N \geq 0$*,* $n \geq 0$*, there exists an initial estimate* $(\hat{a}(0), \hat{b}(0))$ *of the parameter vector such that Algorithm 5.5.3 performed for the set of values* $\{(a^0, b^0), \alpha, (\hat{a}(0), \hat{b}(0))\}$ *involves* n *consecutive controllers which stabilize the true plant* (5.48)*, followed by at least* N *consecutive destabilizing controllers.*

Proof: the proof of Proposition 5.5.4 is based on geometrical considerations. We first spend some words on the geometrical properties of the set of systems leading to a controller stabilizing the true system, what we design by *set of stabilizing systems*.

(1) Set of stabilizing systems in the parameters space: we define the set $\mathcal{S}_{a^0,b^0}$ of stabilizing systems as the set of systems (a, b), $b \neq 0$ such that the control law based on (a, b) according to

$$u(k) = \frac{\alpha + a}{b} y(k) \tag{5.52}$$

stabilizes the system described by (5.48). Hence

$$\mathcal{S}_{a^0,b^0} = \{(a, b) : |\frac{b^0}{b}(a + \alpha) - a^0| < 1\}. \tag{5.53}$$

Equation 5.53 can be geometrically interpreted as follows (see Figure 5.2): define $\mathcal{C}_+$ to be the cone with vertex $(-\alpha, 0)$ and boundaries the two lines going to $(-\alpha, 0)$, $(0, \frac{b^0\alpha}{1+a^0})$ and $(-\alpha, 0)$, $(0, -\frac{b^0\alpha}{1-a^0})$ respectively, such that the elements in $\mathcal{C}_+$ all have a second coordinate $b > 0$. Similarly, define $\mathcal{C}_-$ to be the cone with the same vertex, the same boundaries, but such that the elements in $\mathcal{C}_-$ all have a second coordinate $b < 0$. We then have:

$$\mathcal{S}_{a^0,b^0} = \mathcal{C}_+ \cup \mathcal{C}_-. \tag{5.54}$$

Remark that the line $\mathcal{G}^0$ going through $(-\alpha, 0)$ and (a^0, b^0) represents the set of systems such that when the associated controller is applied to the real system 2.1, the closed-loop pole is exactly in α. And the complement of $\mathcal{S}_{a^0,b^0}$ in $\mathbb{R}^2$ is the set of systems leading to controller which destabilizes the real unknown plant (a^0, b^0). We now recall the following geometrical properties which follow from the orthogonal projection algorithm used in Algorithm 5.5.3.

(2) Orthogonal projection algorithm: geometrically speaking, (5.51) means that the new estimate $\hat{\theta}(k+1)$ is computed as the orthogonal projection of the previous estimate $\hat{\theta}(k)$ on the line given by:

$$\mathcal{G}(k+1) = \{(a, b) \in \mathbb{R}^2 : ay(k) - bu(k) + y(k+1) = 0\}. \tag{5.55}$$

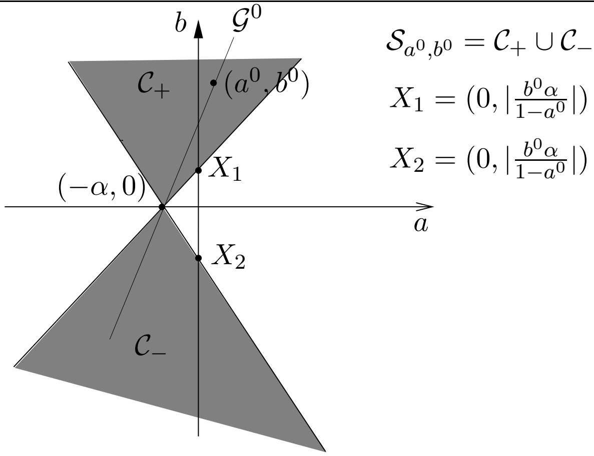


Figure 5.2: Stabilizing controllers.

$\mathcal{G}(k+1)$ has normal vector $\phi(k) = (-y(k), u(k))$ in the parameter space. Now, since $u(k) = \frac{\hat{a}(k)+\alpha}{\hat{b}(k)} y(k))$ (where $\hat{b}(k)$ is supposed to be always non zero), $\mathcal{G}(k+1)$ has normal vector $(-y(k), \frac{\hat{a}(k)+\alpha}{\hat{b}(k)} y(k)$. Hence if $y(k) \neq 0$ and $\hat{b}(k) \neq 0$, the vector $(-\hat{b}(k), \hat{a}(k)+\alpha)$ is normal to $\mathcal{G}(k+1)$ at any time k. This implies that the vector $(\hat{a}(k)+\alpha, \hat{b}(k))$ is parallel to $\mathcal{G}(k+1)$. Let us define $\mathcal{G}^0$ as the line going through $(-\alpha, 0)$ and $\hat{\theta}^0$. It follows from this discussion that $\hat{\theta}(k+1)$ is the orthogonal projection of $\hat{\theta}(k+1)$ on $\mathcal{G}(k)$ parallely to $\mathcal{G}^0$. This result is illustrated in Figure 5.3. Now, the use of the orthogonal projection update rule

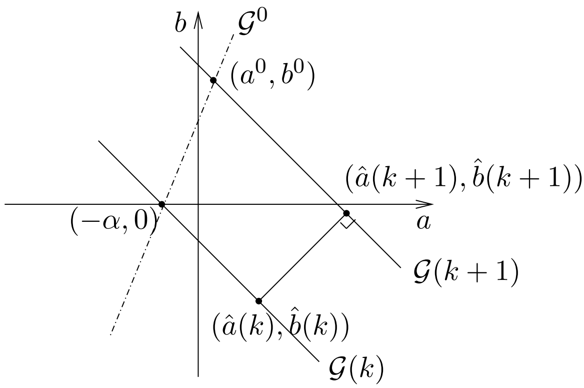


Figure 5.3: Orthogonal projection algorithm.

(5.51) guarantees the following properties.

Property 5.5.5 (Orthogonal projection algorithm)

$$\theta^0 \in \mathcal{G}(k), \forall k. \tag{5.56}$$

$$\lim_{k\to\infty}(\hat{a}(k),\hat{b}(k)) \in \mathcal{G}^0. \quad (5.57)$$

(5.56) implies in particular that $\forall k$, $\hat{\theta}(k+1)$ is the orthogonal projection of $\hat{\theta}(k)$ on the line going through θ^0 and parallel to the line going through $(-\alpha, 0)$ and $(\hat{a}(k),\hat{b}(k))$. This leads to the following result:

$$||(\hat{a}(k+1),\hat{b}(k+1)) - (\hat{a}(k),\hat{b}(k))|| \leq ||(-\alpha,0)-(a^0,b^0)||, \forall k \quad (5.58)$$

This equation means that at any time k, the parameter vector update "step" defined by $||(\hat{a}(k+1),\hat{b}(k+1)) - (\hat{a}(k),\hat{b}(k))||$ is bounded by the *fixed* quantity $||(-\alpha,0)-(a^0,b^0)||$.

Remark 5.5.6 By construction of $(\hat{a}(k),\hat{b}(k))$ (Figure 5.3) we have that the entire sequence of estimates $\{(\hat{a}(k),\hat{b}(k))\}_{k\in\mathbb{N}}$ is located in the half-space with boundary $\mathcal{G}^0$ containing the true parameter vector $(\hat{a}^0,\hat{b}^0)$. This is shown in Figure 5.4.

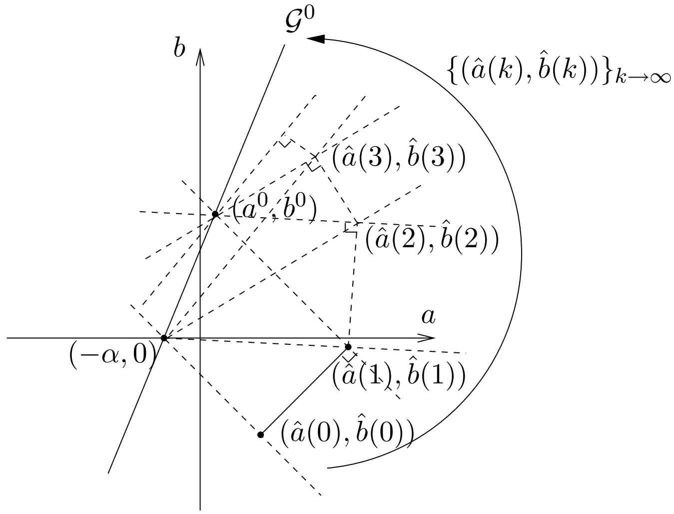


Figure 5.4: Sequence of orthogonal projections.

We now have all the ingredients we need to prove the main result in Proposition 5.5.4. The various steps of this proof are illustrated in Figure 5.5. We fix an integer N, arbitrarily chosen. Suppose that the initial estimate $(\hat{a}(0),\hat{b}(0))$ is on a boundary of the cone $\mathcal{S}_{a^0,b^0}$ of systems yielding a controller stabilizing the actual system. Using Remark 5.5.6, we know that $(\hat{a}(k),\hat{b}(k)) \in\ ]\mathcal{G}^0,(\hat{a}(0),\hat{b}(0))), \forall k \in \mathbb{N}$, where $]\mathcal{G}^0,(\hat{a}(0),\hat{b}(0))$ denotes the open half-plane with boundary $\mathcal{G}^0$ containing $(\hat{a}(0),\hat{b}(0))$. Now, since $\mathcal{G}^0 \in \mathcal{S}_{a^0,b^0}$ (Figure 5.2) and using (5.57), there exists an integer $n_{a^0,b^0} > 0$ such that

$$(\hat{a}(k),\hat{b}(k)) \in\ ]\mathcal{G}^0,(\hat{a}_0,\hat{b}_0))\backslash\mathcal{S}_{a^0,b^0}\ \forall k \leq n_{a^0,b^0}; \quad (5.59)$$

$$(\hat{a}(k),\hat{b}(k)) \in \mathcal{S}_{a^0,b^0}\ \forall k > n_{a^0,b^0}. \quad (5.60)$$

In other words:

$$(\hat{a}(k), \hat{b}(k)) \text{ leads to a controller destabilizing } (a^0, b^0)\ \forall k \leq n_{a^0,b^0}; \tag{5.61}$$

$$(\hat{a}(k), \hat{b}(k)) \text{ leads to a controller stabilizing } (a^0, b^0)\ \forall k > n_{a^0,b^0}. \tag{5.62}$$

Now, by construction (Figure 5.4), we have that

$$\sum_{k=0}^{n_{a^0,b^0}} ||(\hat{a}(k+1), \hat{b}(k+1)) - (\hat{a}(k), \hat{b}(k))|| > ||(a^0, b^0) - (\hat{a}(0), \hat{b}(0))|| \tag{5.63}$$

and

$$\lim_{||(\hat{a}(0),\hat{b}(0))-(-\alpha,0)||\to\infty} [\sum_{k=0}^{n_{a^0,b^0}} ||(\hat{a}(k+1), \hat{b}(k+1)) - (\hat{a}(k), \hat{b}(k))||] = \infty. \tag{5.64}$$

We now recall Equation 5.58:

$$||(\hat{a}(k+1), \hat{b}(k+1)) - (\hat{a}(k), \hat{b}(k))|| \leq ||(-\alpha, 0) - (a^0, b^0)||, \forall k \tag{5.65}$$

Therefore,

$$\sum_{k=0}^{n_{a^0,b^0}} ||(\hat{a}(k+1), \hat{b}(k+1)) - (\hat{a}(k), \hat{b}(k))|| \leq n_{a^0,b^0}||(-\alpha, 0) - (a^0, b^0)|| \tag{5.66}$$

Hence, using equations 5.64 and 5.66, we obtain

$$\lim_{||(\hat{a}(0),\hat{b}(0))-(-\alpha,0)||\to\infty} [n_{a^0,b^0}||(-\alpha, 0) - (a^0, b^0)||] = \infty. \tag{5.67}$$

Since $||(-\alpha, 0) - (a^0, b^0)||$ is a fixed and finite quantity, equation 5.67 is equivalent to

$$\lim_{||(\hat{a}(0),\hat{b}(0))-(-\alpha,0)||\to\infty} n_{a^0,b^0} = \infty. \tag{5.68}$$

These results mean that for any integer N arbitrarily chosen, there exists an initial estimate $(\hat{a}(0), \hat{b}(0))$ taken on the boundary of $\mathcal{S}_{a^0,b^0}$ and far enough from $(-\alpha, 0)$ so that the algorithm leads to at least N destabilizing controllers. To go further, construct the point $(\hat{a}'(0), \hat{b}'(0))$ in such a way that the orthogonal projection of $(\hat{a}'(0), \hat{b}'(0))$ on the line going through (a^0, b^0) and $(\hat{a}(0), \hat{b}(0))$ is $(\hat{a}(0), \hat{b}(0))$ (see Figure 5.5). Note that $(\hat{a}'(0), \hat{b}'(0))$ belongs to $\mathcal{S}_{a^0,b^0}$, i.e., the controller based on $(\hat{a}'(0), \hat{b}'(0))$ stabilizes the real system (a^0, b^0). The classical pole placement algorithm initialized with $(\hat{a}'(0), \hat{b}'(0))$ would hence involve one stabilizing controller (the controller based on $(\hat{a}'(0), \hat{b}'(0))$) followed by at least N consecutive destabilizing controllers. Similarly, for any $n > 0$, we can construct an initial estimate such that Algorithm 5.5.3 initialized with this estimate leads to n consecutive stabilizing controllers followed by at least N consecutive destabilizing controllers. This ends the proof of Proposition 5.5.4.

■

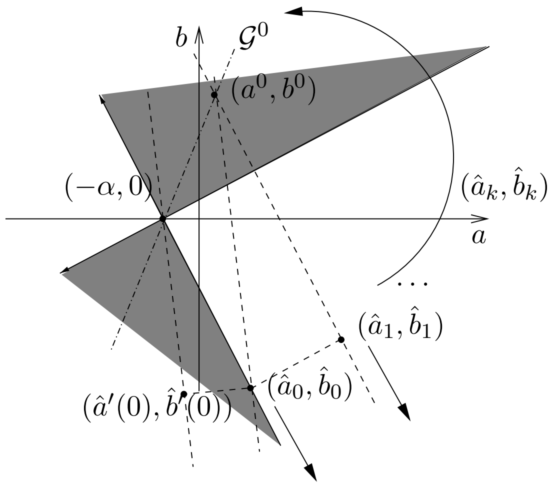


Figure 5.5: Construction of a poor initial estimate.

Remark 5.5.7 Under the constraint that during adaptation the sequence of estimates $\{(\hat{a}(k), \hat{b}(k))\}$ is kept within the region corresponding to asymptotically stable systems $\mathcal{S}_1 = \{(a, b) \in \mathbb{R}^2 : |a| < 1\}$, Proposition 5.5.4 still holds.

It follows from Proposition 5.5.4 that with insufficient prior knowledge on the system to be controlled, classical pole placement might generate arbitrarily poor models, and subsequently arbitrarily poor controllers, leading to bad transients in the input-output response of the closed-loop system. More precisely, Proposition 5.5.4 implies that it is not possible to predict if the classical control system based on Algorithm 5.5.3 will behave badly or not by looking at any arbitrarily large number of initial iterations, since destabilizing controller can be generated at any time of the design. In addition, even in the case where at each frozen time instant the closed-loop system would be obtained, stability of the time-varying system is not necessarily maintained if adaptation is too fast.
In contrast, at no time in our strategy a destabilizing controller is applied to the system to be controlled, even in the case where the initial knowledge on the system is very small, hence no bad transient behavior due to destabilizing controllers can occur and therefore the transient behavior of adaptive systems based on Algorithm 5.5.2 is superior to classical certainty equivalence based schemes. Moreover, once the control phase is started, strong robustness of the model set guarantees that the stability of the closed-loop system is preserved, despite the possibly fast time-variations of the controller. Of course, identification inputs generated in Algorithm 5.5.2 may still have a temporarily destabilizing effect, but this seems to be the inevitable price to be paid due to identification of the initial unknown system.

5.5.3 Simulation example

We now illustrate the ideas discussed in the previous sections by a simulation example. We consider the system defined by (2.1) with

$$a^0 = 0.9;\ b^0 = 5; \quad -\underline{\delta} = \overline{\delta} = 0.1. \tag{5.69}$$

The measurement error $\delta(k)$ is a uniformly distributed random signal with bound $\bar{\delta}$ and with an off-set of value $\bar{\delta}/2$. The control objective is pole placement in $\alpha = -0.3$. The algorithm is initialized with:

$$\hat{a}(0) = -0.3;\ \hat{b}(0) = 0.8;\ y(0) = 2.$$

We compare the performance of the system (5.69) and subject to the three following control pole placement strategies:

- adaptive pole placement based on strong robustness according to Algorithm 5.5.2 using the identification input defined by:

$$u(k) = \gamma_k u_0 \text{ if } k \text{ is even}, \tag{5.70}$$
$$u(k) = \gamma_k u_1 \text{ if } k \text{ is odd}$$

  where the values u_0 and u_1 are:

$$u_0 = 0;\ u_1 = 0.6, \tag{5.71}$$

  and

$$\gamma_k = \sqrt{k+1}, \forall k. \tag{5.72}$$

- classical pole placement given in Algorithm 5.5.3.
- the "true" control input based on the unknown parameters a^0, b^0 given by:

$$u(k) = \frac{\alpha + a^0}{b^0} y(k),\ \forall k. \tag{5.73}$$

The simulation results are depicted in Figure 5.6 and Figure 5.7. In Figure 5.6 the plot of the three control inputs is given, while Figure 5.7 depicts the output responses of the three corresponding control systems. We obtained that after three iterations the uncertainty set identified with the input described in (5.70), (5.71) and (5.72) is strongly robust with respect to pole placement in α. Figure 5.6 and Figure 5.7 show that the performance of the adaptive control system based on strong robustness is better than the performance of the classical adaptive control system, since the transients are improved. Not surprisingly, these transients don't completely vanish. By lack of initial knowledge on the real system, the learning phase indeed requires for a few iterations some input-output signals large enough to achieve strong robustness. Figure 5.8 shows the difference between the optimal control gain in (5.73) and the model-based-control gain sequence of the form

$$\Delta(k) = \frac{\alpha_0 + a^0}{b^0} - \frac{\alpha_0 + \hat{a}(k)}{\hat{b}(k)} \tag{5.74}$$

where the model sequence is obtained from classical adaptive control and adaptive control based on strong robustness respectively. It shows that in this example the control input gain sequence obtained from adaptive control based on strong robustness converges faster to the true gain sequence than the control input sequence obtained with classical adaptive control is. Therefore the control performance is improved by the introduction of strong robustness.

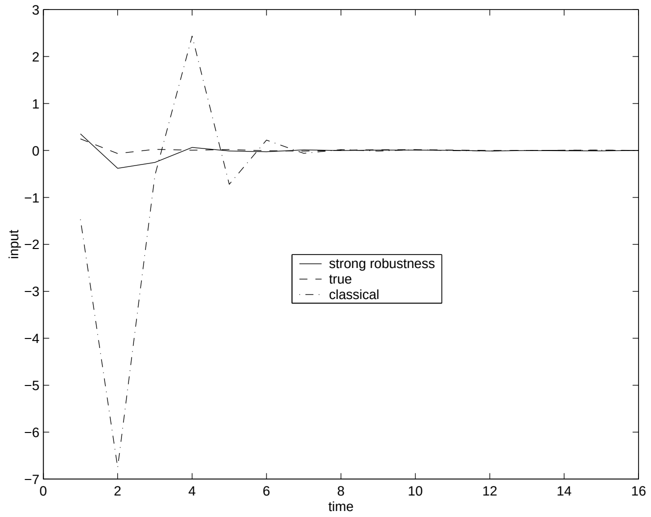


Figure 5.6: Input signal.

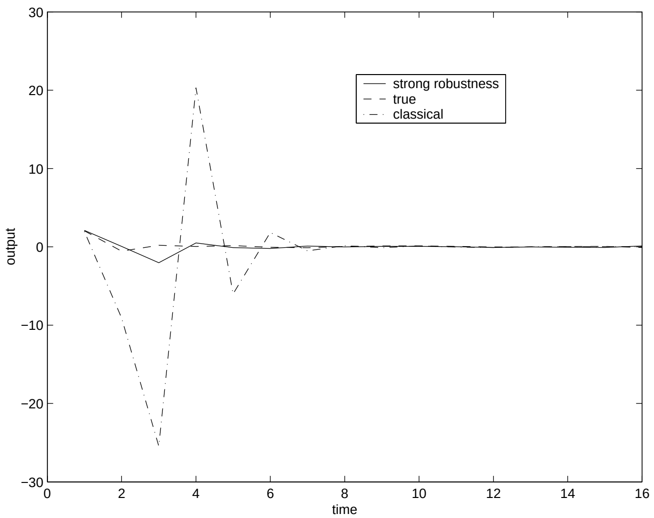


Figure 5.7: Output signal.

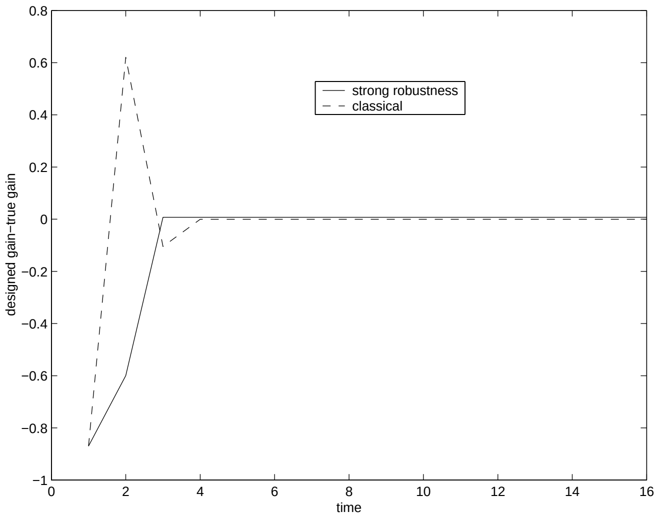


Figure 5.8: Difference between designed feedback gain and true feedback gain.

5.6 Further research

In this section, we briefly discuss how the adaptive control scheme presented in Section 5.3 may be modified. The first potential modification involves time-invariant strong robustness, whereas the second modification includes weak strong robustness.

5.6.1 Time-invariant strong robustness and dwelling time

In the previous sections of this chapter, identification of a strongly robust uncertainty set is the key issue, since actual control can start only after strong robustness has been reached. The main drawback to this is that the test which allows us to test whether the uncertainty set is strongly robust or not given in Theorem 5.3.8 has a complexity which grows very fast with the considered system order. Hence we find appealing the idea of trying to decrease the computational complexity, at least in a first time period. To this respect, we first recall that in order to be strongly robust, a given set of systems must necessarily be time-invariant strongly robust (see Definition 3.1.10): the controller based on any model in this set has to stabilize any other system in the set. Time-invariant strong robustness is already a stringent condition on the model set, however to test whether a set is time-invariant strongly robust may be computationally more tractable than to test whether it is strongly robust (Subsection 3.3.4, Chapter 3). This suggests therefore to split the identification phase described in Section 5.3.1 into two phases: at first we would collect information on the system to be controlled until the uncertainty set is time-invariant strongly robust. This condition on the uncertainty set is already stringent but is satisfied in finite and reasonable time when using the input sequence (5.11)-(5.14). To this respect, it has been proven in a slightly different context that stabilizing the identified class of models automatically leads to stabilization of the true unknown system [32], [97].

Once it has been checked that time-invariant strong robustness is achieved, one may proceed according one of the two following points of view. First time-variations of the controller should also be taken into account so that strong robustness is achieved: it should be checked that the time-varying controller based on any sequence of systems in the uncertainty set stabilizes any fixed system in the set. This could be done by checking if the uncertainty set satisfies the condition (5.27) at any time. This method, where checking time-invariant strong robustness is then refined into a test checking strong robustness, is a way to adapt the effort put in the identification procedure to the desired level of information: time-invariant strong robustness is less constraining than strong robustness and is easier to handle from a computational point of view.
Alternatively, one may already start control using adaptation of a model and certainty equivalence, while checking at any time that the stability of the time-varying closed loop system is not disrupted. More precisely, at each time we would estimate a model according to (5.29), compute the controller on the basis of this estimate according to (5.30), but at the same time force the time-variations of the controller to be mild enough so that asymptotic stability of the overall scheme is preserved. Such an idea suggests to introduce a so-called *dwelling time* [32] between consecutive instants at which the model is updated, in such a way that it would be adaptively selected on the basis of collected data measurements. At the same time, one could keep checking whether the uncertainty set becomes strongly robust or not; if strong robustness is achieved, then the dwelling time could be put to zero since asymptotic stability would be secured, irrespectively of the time-variations of the controller. However, how to compute adaptively such a dwelling time in our framework is not clear yet and requires further investigation.

5.6.2 Adaptive control and weak strong robustness

In Chapter 3, we introduced the notion of weak strong robustness as follows: a set $\Omega \subset \mathcal{C}_n$ is weakly strongly robust if there exists a control objective satisfying Assumption 5.2.2 in a class of candidate control objectives such that Ω is strongly robust with respect to this control objective. Now, suppose that we deal with adaptive pole assignment, i.e., suppose that the adaptive control objective is to obtain closed-loop poles $\alpha_i(k)$, $i = 1, \cdots, 2n-1$ that are asymptotically equal to desired fixed stable poles α_i, $i = 1, \cdots, 2n-1$. In this situation, one may compute at each time the set of pole locations $\{\alpha_1^k, \cdots, \alpha_{2n-1}^k\} \subset (]-1, 1[)^{2n-1}$ with respect to which the set $\hat{\mathcal{G}}(k)$ would be strongly robust: if this set is not empty, i.e., if $\hat{\mathcal{G}}(k)$ is weakly strongly robust, then by comparing the position of desired pole locations to the location of this set might shed some light on how the model set $\hat{\mathcal{G}}(k)$ should be updated so that at the next iteration, the desired poles are located in the set of poles for which $\hat{\mathcal{G}}(k+1)$ is strongly robust. Such method would indeed provide a way to minimize the time needed to the identification of a strongly robust uncertainty set in the algorithm presented in Section Section 5.3.
Although the practical application of this idea is not clear in the general case, it yields interesting results in the case of first order systems. Given a bounded set of systems $\hat{\mathcal{G}}(k) \subset \mathcal{P}_1$, it is easy to compute the set of pole locations denoted by $[\alpha_m, \alpha_M]$ for which $\hat{\mathcal{G}}(k)$ is strongly robust. Geometrically, the smallest pole value α_m for which $\hat{\mathcal{G}}(k)$ is strongly robust, if it exists, is given by the intersection of the parallel line to the tangent to $\hat{\mathcal{G}}(k)$ going through $(1, 0)$ with the $a-$axis. Similarly, the largest pole value α_M for which $\hat{\mathcal{G}}(k)$ is strongly robust, if

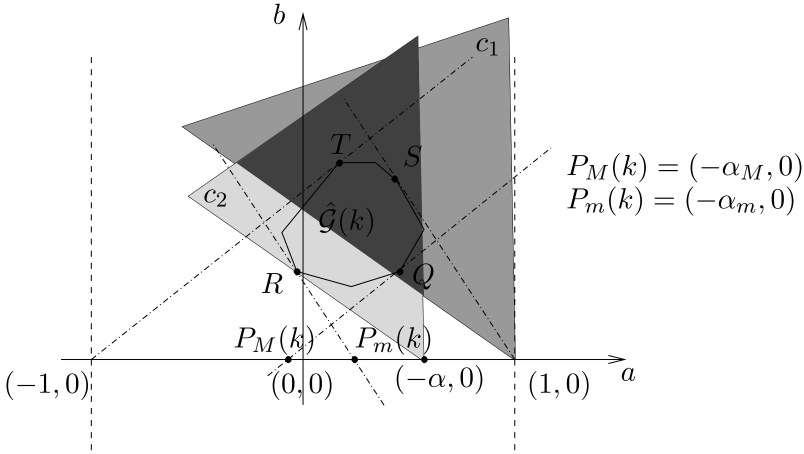


Figure 5.9: Weak strong robustness and update of the model set.

it exists, is given by the intersection of the parallel line to the tangent to $\hat{\mathcal{G}}(k)$ going through $(-1, 0)$ with the $a-$axis.

If $-1 < \alpha_m < \alpha_M < 1$ as depicted in Figure 5.9, then $\hat{\mathcal{G}}(k)$ is strongly robust with respect to pole placement in any pole within $[\alpha_m, \alpha_M]$. Now, suppose that the desired pole is α, $|\alpha| < 1$.

1. If $\alpha \in [\alpha_m, \alpha_M]$, then $\hat{\mathcal{G}}(k)$ is strongly robust with respect to the desired pole placement, hence the control phase can be started.

2. If $\alpha_M < \alpha < 1$, then $\hat{\mathcal{G}}(k)$ is not strongly robust with respect to the desired pole placement. A way to see non-strong robustness is that the intersection of the cones c_1 and c_2 is non-empty (see Chapter 3). Moreover, we see from the above geometrical consideration that $\hat{\mathcal{G}}(k)$ can never become strongly robust if the systems corresponding to R, S in Figure 5.9 belongs to $\hat{\mathcal{G}}(k)$. Hence, the identification could be forced so as to cut the points R, S off the uncertainty set.

3. If $-1 < \alpha < \alpha_m$, then $\hat{\mathcal{G}}(k)$ is not strongly robust with respect to the desired pole placement. Moreover, we see from the previous geometrical considerations that $\hat{\mathcal{G}}(k)$ can never become strongly robust if the systems corresponding to T, Q in Figure 5.9 belongs to $\hat{\mathcal{G}}(k)$. Hence, the identification could be forced so as to cut the points T, Q off the uncertainty set.

Thus the position of the desired closed-loop poles with respect to the set of stable poles for which the uncertainty set would be strongly robust would inform us about in which direction the uncertainty set should be shrunk so as to become strongly robust as fast as possible.

5.7 Conclusions

In this chapter, the results of Chapter 3 and Chapter 4 have been exploited to revisit classical adaptive control of linear time-invariant SISO systems in discrete-time with an unknown-but-bounded uncertainty. Yet the tests proposed to check the controllability and strong robustness conditions in our approach are only sufficient, hence results are conservative. However we ensure that the multi-phase adaptive scheme based on strong robustness will in finite time perform the control of the unknown plant to be controlled. The analysis shows that bad transient cannot occur, in opposition to classical schemes where destabilizing controllers are a priori not avoided. Of course, the approach proposed in this chapter is a general description and still open questions remain. For instance, note that we expect the test for strong robustness to be the more expensive task in terms of computation. Also, it might be interesting to compute a test for strong robustness that would be recursive. Such a test would spare us with rechecking the test for strong robustness over the whole set of model candidates at each new measurement, and the computational cost of the approach would be much lower.

Chapter 6

Conclusions and further research

In this survey chapter, we first summarize the preceding chapters so as to point out their main contribution. Efficiency and necessity of the strongly robust adaptive control methodology are discussed, as well as the limitations that may be encountered when resorting to this approach. Some of these limitations are related to the numerical tools used to tackle the problems involved in the presented algorithm and could possibly vanish if other mathematical or conceptual tools were used. On the other hand, some of the limitations are inherent to the strong robustness approach, and cannot be reduced unless by considering other adaptive control techniques. These two issues lead to our recommendations for further research.

6.1 Conclusions

In Chapter 1, the general context of the thesis, that of adaptive control, has been presented. Classically, adaptive control approaches are derived from the certainty equivalence principle, as it is briefly outlined next. First, identification methods deliver an approximation of the plant (the model) and a level of accuracy of this model (the uncertainty). Second, based on the model, a controller is designed to be applied to the real plant to be controlled. Clearly, the performance achieved by this model-based controller highly depends on the quality of the model but also on the assumed uncertainty. This is the reason why when control performance is not considered as good enough, new measurement data are used to identify a new model, allowing the update of the model-based controller. This idea of adapting the model until performance is satisfactory is the key idea governing adaptive control. It has been shown in the literature (see Chapter 1) that most of certainty equivalence-based adaptive control strategies yield control design in the sense that the controlled system will perform well asymptotically. This is because the model is updated in such a way that it becomes asymptotically good for control. However, in the initial phase, when model uncertainty is large, there is no guarantee that the model-based controller performs well when applied to the real system. An undesired but no predictable case is when the model-based controller does not stabilize the true system. Or, in an even more critical situation, the model could be uncontrollable. In addition, due to adaptation, time variations of the controller may destroy asymptotic stability of the control system. These three phenomena might lead to undesired transients or loss of stability, and are therefore highly undesired. Hence the question: *How can one arrive at a high performance closed-loop controlled plant on the basis of plant models that are validated by measurement data, whilst insuring stability of this controlled plant at any time?* Our solution to this challenging question is the main object of this thesis. To tackle this problem, we reformulate it as follows: ”What property should the set of all model candidates satisfy so that the three drawbacks stated above vanish when using classical certainty equivalence adaptive control methods?” The answer to this question is that the model set has to be strongly robust. If strong robustness is achieved, then the time-varying model-based controller exists and stabilizes the plant to be controlled, at any time of the design.

In Chapter 2, the mathematical set-up has been described. The system to be studied is a linear, time-invariant and controllable SISO system with known order described in discrete-time. Moreover, the modeling error is bounded-but-unknown, with known lower and upper bounds. The control objective is left unspecified; however the map assigning to each model in the model class its controller is continuous and it is assumed that the closed-loop system obtained when connecting any model and its corresponding controller is asymptotically stable.

In Chapter 3, the notion of strong robustness has been treated as a mathematical property of a set of systems. First, the definition of strongly robust sets of systems in the class of systems presented in Chapter 2 has been given. Then, various notions related to strong robustness have been defined: time-invariant strong robustness, weak strong robustness and strong quadratic robustness. An important result in this chapter has been the proof that around any system in the class of systems defined in Chapter 2, there exists an open strongly robust neighborhood. The introduced strong robustness notions have been illustrated by means of first order case examples. Furthermore, relationship between these presented notions and classical robustness has been established. In particular, strong robustness measures have been expressed by means of real and complex structured stability radii. This allowed us to derive sufficiency criteria for the strong robustness notions listed above involving structured stability radii. However, to verify numerically whether a given set of systems satisfies such tests is not trivial. In order to deliver a computationally tractable test for strong robustness, attention has been then paid to polyhedral sets of systems in canonical form in the class of systems specified in Chapter 2, in the case of pole placement design. Under these assumptions, a necessary and sufficient test for strong quadratic robustness has been expressed under the form of a finite set of Linear Matrix Inequalities. Next, a Kharitonov-like criterion to test whether a given set of systems is time-invariant strongly robust for pole placement design has been established.

In Chapter 4, an input design to identify a strongly robust set of models has been presented. This input sequence is chosen to be $2n$-periodic, where n is the order of the system to be controlled. To begin with, the case where the output sequence is also $2n$-periodic is considered and conditions on the $2n$ design parameters are established to ensure boundedness and decreasing size of the uncertainty set. Then, these results have been extended to the non-periodic case, and the design of an input sequence yielding a strongly robust uncertainty set in finite time has been explicitly given. Finally, the effectiveness of this designed input sequence in terms of decrease of the size of the uncertainty set with time has been shown on a first order example.

In Chapter 5, identification of a strongly robust uncertainty set and adaptive control have been brought together, leading to strongly robust adaptive control. After having described this new adaptive control approach, its analysis has been provided. It has been shown that undesired transients cannot occur when resorting to strongly robust adaptive control, contrary to classical approaches where arbitrarily large transients may appear. Although our method does not require the knowledge of the exact parameters of the system to be controlled, controllability of the model and stabilizability of the model-based controller are guaranteed and stability of the control scheme is preserved irrespective of the speed of adaptation. The overall scheme is illustrated by means of a first order example.

6.2 Recommendations for further research

Many questions related to the strong robustness approach depicted in this thesis remain open. Limitations of the proposed approach are now examined and potential relaxation of these constraints after further investigation is now discussed.

6.2.1 Can we relax the standing assumptions?

The results presented in this thesis have been established under the assumptions presented in Chapter 2. Below, we investigate whether potential modifications of this new approach may allow us to relax these assumptions.

- We assumed all along this work that the system to be controlled is open-loop asymptotically stable. This assumption is required for open-loop identification as discussed in Chapter 3. However, if identification of a strongly robust uncertainty set could be achieved by closed-loop identification, this assumption could be relaxed. On the other hand, due to the complex interaction between identification and control in closed-loop systems, identifiability problems may occur and it is not established yet how closed-loop identification of a strongly robust model set could be performed. Further investigation in this line of thought may be fruitful.

- Throughout this thesis, the order of the system to be controlled is assumed to be known. In particular, the identification input design proposed in Chapter 4 tightly depends on this assumption since it deals with $2n$-periodic input sequences, where n is the assumed system order. Now, one may desire to weaken the assumption that the exact system order is known. For instance we may assume that only an upper bound on this order is available, say, $\overline{n} \geq n$. Note that the definition of strongly robust sets of systems given in Chapter 3 still applies to sets of systems that have different orders. Now, to deal with adaptive control of a system with unknown order, how to modify our strongly robust adaptive control method in Chapter 5? One answer to this question may be to resort to a set of strongly robust adaptive control algorithms run in parallel, each of these algorithms being deviced for a certain order value. It is not established yet how such an idea could be developed but this would certainly be a nice solution, at least in the case where a known upperbound of the true system order is known and not too large.

- In this thesis, the systems are described in discrete-time. However, all the presented results can easily be mimicked to the continuous-time description. Characterization of strongly robust sets of systems in continuous time description could then be expressed by means of Riccati equations, following the approach in Chapter 3 involving LMI's.

- It should be emphasized that the definition of strongly robust sets (and other related strong robustness notions) can be extended to a much broader class of systems than the class of systems defined in Chapter 2. In particular, time-varying systems could be considered, leading to strongly robust sets of time-varying systems. In the same line of thought, the case of nonlinear Multi-Input Multi-Output (MIMO) systems may be investigated. However, it is far from clear how to compute the region of all model candidates on the basis of data measurements when the system to be controlled is MIMO and presents time-variations or nonlinearities in its dynamics. Moreover, existence of strongly robust sets of systems within this much broader class of systems is not guaranteed and probably would require further assumptions on the considered systems.

6.2.2 Test for strong robustness: conservatism issue

As discussed in Chapter 3, only sufficiency tests to secure strong robustness have been established up to this date, and these tests may hence be conservative. A first, very natural, question to ask is: how conservative are these tests? This problem has not been examined in this thesis but would probably shed some light on when the proposed approach is inappropriate or, on the contrary, very much advised.
Moreover, as a result of the strong robustness test conservatism, it may happen that the identified uncertainty set is strongly robust while the sufficient test for strong robustness is not satisfied. In such a case, more identification steps would be required before control can be started, although control could be theoretically started earlier. To alleviate this problem, further work should hence deliver a sufficient and necessary test to check whether a given set of systems is strongly robust or not.

6.2.3 Do we have to wait for strong robustness to start control?

In the strong robustness adaptive control approach, validation of the test checking strong robustness of the identified uncertainty set is the criterion which decides when control can actually start. However, this test has a complexity which grows very fast with the system order. Moreover, due to the conservatism of this test, the time at which control actually starts might be very large, although strong robustness may have been achieved at an earlier time. In addition, strong robustness may be achieved only when the uncertainty set is very small, requiring a large number of measurements. Hence the question: "Is there a time at which control can be started under appropriate precautions, although the test for strong robustness is not yet validated, instead of waiting until this test is validated?" In order to be strongly robust, a given set of systems must be time-invariant strongly robust. Time-invariant strong robustness is already a stringent condition on the model set, however to test whether a set is time-invariant strongly robust may be computationally more tractable than to test whether it is strongly robust. This suggests therefore to split the identification phase described in Section 5.3.1 into two phases: first, we would collect information on the system to be controlled until the uncertainty set is time-invariant strongly robust. Once it has been checked that time-invariant strong robustness is achieved, one may already start control using adaptation of a model and certainty equivalence, while checking at any time that the stability of the time-varying closed loop system is not disrupted by adaptation. Such an idea suggests to introduce a so-called *dwelling time* between consecutive instants at which the model is updated, to be adaptively selected on the basis of collected data measurements (see Section 5.6.1 and Chapter 5 for references to recent contributions involving this idea). This dwelling time should be kept small enough to guarantee that the adaptation process would not destroy stability of the closed-loop system. At the same time, one could keep checking whether the uncertainty set becomes strongly robust or not; if strong robustness is achieved, then the dwelling time could be put to zero since asymptotic stability would be secured, irrespectively of the time-variations of the controller. However, how to compute adaptively such a dwelling time in our framework is not clear yet and requires further investigation.

6.2.4 How data can serve identification for strong robustness?

In Chapter 5, it appeared that the way the uncertainty set should be reduced geometrically may give a hint as to how to choose the identification sequence so that strong robustness is achieved as fast as possible. Also, the location of the desired closed-loop poles with respect to the set of closed-loop poles that are admissible for strong robustness may inform the designer on how much and how fast the uncertainty set should be shrunk so as to achieve strong robustness. These ideas, clearly established in the case of first order pole placement (Section 5.6.2), are still far from trivial for larger order systems. In particular, one interesting question is: What kind of geometry have strongly robust sets of systems? Unfortunately, this simple question cannot be answered at a complete level of generality. For instance, if the true plant to be controlled is squeezed towards the set of non-controllability systems, one expects the largest strongly robust set of systems containing this plant to shrink. Further investigation may relate the size of the largest strongly robust neighborhoods around a system to the level of controllability of this system.

6.2.5 When to use strongly robust adaptive control?

The main drawback of our approach is that it involves computationally expensive steps, such as the computation of the membership set, the test to check whether this set is in the set of controllable systems and the test to check whether strong robustness is achieved or not. Hence, there may be some situations where one should certainly think twice before using strongly robust adaptive controllers. As such, in the cases where poor-quality transients are not a very serious problem for applications, one may instead resort to classical robust adaptive control methods leading to much simpler controllers. However, it is important to note that when prior knowledge is not sufficient to guarantee good transients, and when bad transients are absolutely undesired, classical adaptive control methods may fail. If strongly robust adaptive control is adopted, more effort has to be put in the identification part and the time at which control of the system will actually start may be large, but no risk of bad transients will ever occur. On the other hand, if classical adaptive control is preferred then control starts earlier, but without any guarantee to keep transients acceptable nor guaranteeing that closed-loop stability will be secured during the adaptation process.

Bibliography

[1] H. Akcay, The size of the membership-set in a probabilistic framework, *Proc. 13th IFAC Symposium on System Identification, Rotterdam, The Netherlands*, 2003.

[2] H. Ackay and P. P. Khargonekar, The least squares algorithm, parametric system identification and bounded noise, *Automatica*, 39: 1535-1540, 1993.

[3] D. Amir, J. Mach and K. Saatkamp, Existence of Chebyshev centers, best n-sets and best compact approximations, *Trans. Amer. Math. Soc.*, 271:513-524, 1982.

[4] B. D. O. Anderson, R. R.. Bitmead, C. R. Johnson, Jr., P. V. Kokotovic, R. L. Kosut, I. M. Y. Mareels, L. Praly and B. D. Riedle, *Stability of Adaptive Systems: Passivity and Averaging Analysis*, Cambridge, MA: M.I.T. Press, 1986.

[5] A. C. Antoulas and B.D.O. Anderson, On the choice of inputs in identification for robust control, *Automatica*, 35: 1009-1031, 1999.

[6] K. Arent, *Selected Topics in Adaptive Control*, Ph.D. Thesis, Dept. of Applied Mathematics, The University of Twente, Enschede, The Netherlands, 1995.

[7] K. J. Åström and B. Wittenmark, *Adaptive control*, Second Edition, Addison-Wesley Series in Electrical Engineering: Control Engineering, 1995.

[8] E. W. Bai, M. Fu, R. Tempo and Y. Ye, Convergence Results of the Analytical Center Estimator, *IEEE Transactions on Automatic Control*, 45(3): 569-572, 2000.

[9] E. W. Bai, K. M. Nagpal and R. Tempo, Bounded-error Parameter Estimation: Noise Models and Recursive Algorithms, *Automatica*, 32: 985-999, 1996.

[10] E. W. Bai, L. Qiu and R. Tempo, Unfalsified Weighted Least Squares Estimates in Set-Membership Identification, *IEEE Trans. on Circuits and Systems I: Fundamental Theory and Applications*, 45(1): 41-49, 1998.

[11] E. W. Bai, R. Tempo and H. Cho, Membership Set Estimators: Size, Optimal Inputs, Complexity and Relations with Least Squares, *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 42(5): 266-277, 1995.

[12] E. W. Bai, Y. Ye and R. Tempo, Bounded Error Parameter Estimation: A Sequential Analytic Center Approach, *Proc. of the 36th IEEE CDC, San Diego*, 732-737, 1997.

[13] S. Barnett, *Polynomials and Linear Control Systems*, Marcel Dekker, New York, 1983.

[14] Y. Bar-Shalom and E. Tse, Dual effect, certainty equivalence, and separation in stochastic control, *IEEE Transactions on Automatic Control*, 19:494-500, 1974.

[15] G. Belforte, Uncertainty evaluation for estimates from recursive projection algorithms, *International Journal of Adaptive Control and Signal Processing*, 9(1): 97-106, 1995.

[16] G. Belforte, B. Bona and V. Cerone, Identification, structure selection and validation of uncertain models with set-membership error description, *Mathematics and Computer in Simulation*, 32: 561-569, 1990.

[17] G. Belforte and M. Milanese, Uncertainty intervals evaluation in presence of unknown but bounded errors- Nonlinear families of models, *Proc. of the 1st IASTED Int. Symposium on Modeling, Identification and Control*, Davos, Switzerland, 1981.

[18] R. R. Bitmead, *Adaptive Optimal Control: the Thinking Man's GPC*, Prentice Hall, 1990.

[19] R. G. Bland, D. Godfarb and M. J. Todd, The ellipsoid method: A survey, *Operations Research*, 29(6):1039-1091, 1981.

[20] D. L. Boley and W. S. Lu, Measuring how far a controllable system is from an uncontrollable one, *IEEE Transactions on Automatic Control*, 31: 249-252, 1986.

[21] D. Borelli, E. Mosca and J. Zhang, Sidestepping the singularity problem via self-excitation in adaptive 2-DOF control, *Proc. of the 3rd European Control Conference*, 885-890, 1995.

[22] N. K. Bose and J. F. Delansky, Tests for robust Schur interval polynomials, *30th Midwest Symposium on Circuits and Systems*, G. Glasford and K. Jabbour, Eds., Elsevier Sciences Publ., 1357-1361, 1988.

[23] S. P. Boyd and C. H. Baratt, *Linear Controller design: limits of performance*, Prentice Hall Information and System Science series, 1991.

[24] R. Byers, Detecting nearly uncontrollable pairs, *Proc. in Numerical Methods, Proc. of the Int. Symposium MTNS-89*, Amsterdam, 3:447-457, 1990.

[25] M. Cadic and J. W. Polderman, Strong robustness in adaptive control, *LNCIS 281, Nonlinear and Adaptive Control NCN4 2001*, Springer, 45-54, 2003.

[26] M. Cadic and J. W. Polderman, Strong robustness of boxes of systems: a Kharitonov-like test, *Proc. 8th Mechatronics Forum International Conference 'Mechatronics'*, Enschede, The Netherlands, 2002.

[27] M. Cadic, J.W. Polderman, Strong Robustness in Multi-Phase Adaptive Control: the basic Scheme, *Proc. MTNS 2002*, Indiana, USA, also in *International Journal of Adaptive Control and Signal Processing*, 2003, to appear.

[28] M. Cadic, J.W. Polderman and I.M.Y Mareels, Set membership identification for adaptive control: input design, *Proc. 42nd IEEE Conference on Decision and Control, Hawaii, USA*, 2003.

[29] M. Cadic, S. Weiland and J. W. Polderman, Strong robustness measures of linear SISO systems, *Proc. of the 13th IFAC Symposium on System Identification, SYSID 2003*, Rotterdam, The Netherlands, 2003.

[30] M. C. Campi, The problem of pole-zero cancellation in transfer function identification and application to adaptive stabilization, *Automatica*, 32:849-857, 1996.

[31] M. C. Campi, Adaptive control of non-minimum phase systems, *International Journal of Adaptive Control and Signal Processing*, 9:137-149, 1995.

[32] M. C. Campi, J. P. Hespanha and M. Prandini, Cautious hierarchical switching control of stochastic linear systems, *International Journal of Adaptive Control and Signal Processing*, 2003, to appear.

[33] M. C. Campi and P. R. Kumar, Optimal Adaptive Control of an LQG system, *Conference on Decision and Control, Kobe, Japan*: 349-353, 1996.

[34] M. F. Cheung, S. Yurkovich and K. M. Passino, An optimal volume ellipsoid algorithm for parameter set estimation, *IEEE Trans. on Automatic Control*, 38: 1292-1296, 1993.

[35] R. Cristi, A note on indirect adaptive control with stabilizable plant estimates, *IEEE transactions on Automatic Control*, 32:1039-1040, 1987.

[36] J. Demmel, On condition numbers and the distance to the nearest ill-posed problem, *Numer. Math.*, 51(3):251-289, 1987.

[37] J. Doyle, Analysis of feedback systems with structured uncertainties, *Proc. IEE. 129*, 242-250, 1982.

[38] R. Eising, Between controllable and uncontrollable, *Systems & Control Letters*, 4:263-264, 1984.

[39] E. Fogel and Y. F. Huang, On the value of information in system identification: bounded noise case, *Automatica*, 18(2): 229-238, 1982.

[40] P. A. Fuhrmann, A polynomial approach to linear algebra, *Springer Verlag, New York*, p. 63, 1996.

[41] D. Godfarb and M. J. Todd, Modifications and implementation of the ellipsoid algorithm for linear programming, *Mathematical Programming*, 23: 1-19, 1982.

[42] G. C. Goodwin and R. L. Payne, Dynamic system identification: Experiment design and data analysis, Academic Press, New York, 1977.

[43] G. C. Goodwin and K. S. Sin, *Adaptive filtering prediction and control*, Prentice Hall, Englewood Cliffs, NJ, 1984.

[44] M. Gu, New Methods for Estimating the Distance to uncontrollability, *SIAM J. Matrix Anal. Appl.*, 21(3):989-1003, 2000.

[45] S. Gugercin, A. C. Antoulas and H.P. Zhang, The issue of consistency in identification for robust control, *Automatica*, Technical report, 2000.

[46] P.-O. Gutman, Robust and adaptive control: fidelity or an open relationship?, *Systems & Control Letters*, 49(1):9-19, 2003.

[47] C. He, Estimating the distance to uncontrollability: A fast method and a slow one, *Systems & Control Letters*, 26:275-281, 1995.

[48] J. P. Hespanha, D. Liberzon and A. S. Morse, Overcoming the limitations of adaptive control by means of logic-based switching, *Systems & Control Letters*, 49:49-65, 2003.

[49] J. P. Hespanha and A. S. Morse, Stability of switched systems with average dwell-time, *Proc. of the 38th IEEE Conf. on Decision and Control*, Phoenix, AZ, 2655-2660, 1999.

[50] D. Hinrichsen, B. Kelb and A. Linnemann, An algorithm for the computation of the complex stability radius, *Automatica*, 25:771-775, 1989.

[51] D. Hinrichsen and A. J. Pritchard, Robustness measures for linear state space systems under complex and real parameter perturbations, *Perspectives in Control Theory, Proc. of the Sielpia Conference*, Sielpia, Poland, Birkhäuser, 1988.

[52] D. Hinrichsen and A. J. Pritchard, Stability radii of linear systems, *Systems & Control letter*, 7:1-10, 1986.

[53] J. Hoefkens and M. Berz, Verification of Invertibility of Complicated Functions over Large Domains, *Reliable Computing*, Kluwer Academic Publishers, Boston, 8:1-16, 2002.

[54] P. A. Ioannou and J. Sun, *Robust Adaptive Control*, Prentice Hall, 1996.

[55] P. A. Ioannou and A. Datta, Robust adaptive control: design analysis and robustness bounds, *Foundation of Adaptive Control*, P. V. Kokotović (Ed.), New York, Springer-Verlag, 1991.

[56] H. Kaufman, I. Barkana and K. Sobel, *Direct Adaptive Control Algorithms, Theory and Applications*, Second Edition, Springer-Verlag New York, 1998.

[57] V. L. Kharitonov, The Routh-Hurwitz problem of a family of polynomials and quasipolynomials, *Izvetiy Akademii Nauk Kazakhskoi SSR, Seria fizikomatematicheskaia*, 26:69-79, 1979.

[58] H. Konig and D. Pallashke, On Khachian's algorithm and minimal ellipsoids, *Numer. Math.*, 36: 211-223, 1931.

[59] R. L. Kosut, *Model Identification and Adaptive Control: From Windsurfing to Telecommunications*, G. Goodwin (ed.), Springer-Verlag, 2001.

[60] F. Kraus, M. Mansour and E. I. Jury, Robust Schur-stability of interval polynomials, *IEEE Transactions on Automatic Control*, 37(1):141-143, 1992.

[61] G. Kreisselmeier, A robust indirect adaptive control approach, *Int. Journal Cont.*, 43:161-175, 1986.

[62] G. Kreisselmeier and M. C. Smith, Stable adaptive regulation of arbitrary n-th order plants, *IEEE Transaction on Automatic Control*, 31:299-305, 1989.

[63] P. R. Kumar, Convergence of adaptive control schemes using least-squares estimates, *Proc. of the 28th IEEE Conference on Decision and Control*, Tampa, USA, 727-731, 1989.

[64] P. de Larminat, On the stabilizability condition in indirect adaptive control, *Automatica*, 20:793-795, 1984.

[65] J. Lelong-Ferrand and J. M. Arnaudiès, *Cours de mathématiques, Tome 1, Algèbre, 3rd Ed, Classes préparatoires, Bordas, Paris*, 1978.

[66] T. M. Lin, M. Nayeri and J. R. Deller, A consistently convergent OBE algorithm with automatic estimation of error bounds, *Int. Journal of Adaptive Control and Signal Processing*, 12(4): 305-324, 1998.

[67] L. Ljung, *Systems Identification: Theory for the User*, Prentice Hall, Englewood Cliffs, NJ, 1987.

[68] R. Lozano, J.-M. Dion and L. Dugard, Singularity-free adaptive pole placement without resorting to persistence of excitation detailed analysis for first order systems, *Automatica*, 28:27-33, 1992.

[69] R. Lozano and X. H. Zhao, Adaptive pole placement without excitation probing signals, *IEEE Trans. Automat. Control*, AC-39:47-58, 1994.

[70] L. Luo and D. E. Miller, Near optimal LQR Performance for Uncertain First Order Systems, *Int. J. Adapt. Control Signal Process.*, 2003, to appear.

[71] I. M. Y. Mareels, R. R. Bitmead, M. Gevers, C. R. Johnson and R. L. Kosut, How exciting can a signal really be?, *Systems and Control Letter*, 8, 197-204.

[72] I. M. Y. Mareels, J. W. Polderman, *Adaptive Systems, An Introduction*, Systems & Control: Foundations & Applications, Birkhäuser, 1996.

[73] J. M. Martin, State space measures for stability robustness, *IEEE Transactions on Automatic Control*, AC-32:509-512, 1987.

[74] J. M. Martin and G. A. Hewer, Smallest destabilizing perturbations for linear systems, *International Journal of Control*, 45:1495-1504, 1987.

[75] S. P. Meyn and P. E. Caines, The zero divisor problem of multivariable adaptive control, *Systems & Control Letters*, 6:235-238, 1985.

[76] R. H. Middleton, G. C. Goodwin, D. J. Hill and D. Q. Mayne, Design issues in adaptive control, *IEEE Trans. Automat. Control*, AC-33:50-58, 1988.

[77] M. Milanese, Properties of least squares estimates in set-membership identification, *Automatica*, 31: 327-332, 1994.

[78] M. Milanese, J. Norton, H. Piet-Lahanier and E. Walter, Eds, *Bounding approaches to system identification*, London, Plenum, 1996.

[79] M. Milanese and A. Vicino, Optimal estimation theory for dynamic systems with set-membership identification: an overview, *Automatica*, 27: 997-1009, 1991.

[80] M. Milanese and A. Vicino, Estimator theory for nonlinear models and set-membership uncertainty - An overview, *Automatica*, 27: 997-1009, 1991.

[81] S. H. Mo and J. P. Norton, Fast and robust algorithm to compute exact polytope parameter bounds, *Mathematics and Computers in Simulation*, 32(5-6): 481-493, 1990.

[82] E. Mosca and J. Zhang, Robust adaptive predictive control with self-excitation, *Proc. 32nd Conf. on Decision and Control*, 576-581, 1993.

[83] M. Nordin, P.-O. Gutman, Non-Linear speed control of elastic systems with backlash, *Proc. of the 39th IEEE Conf. On Decision and Control*, Sydney, 12-15:4060-4065, 2000.

[84] J. P. Norton and S. H. Mo, Parameter bounding for time-varying systems, *Mathematics and Computers in Simulation*, 32: 527-534, 1990.

[85] U. Nurges, On discrete Kharitonov theorem and robust controller design, *Proc. IMACS Multiconference on Computational Engineering in Systems Application*, Nabeul-Mammamet, Tunisia, 1:495-499, 1998.

[86] U. Nurges and E. Rüstern, On the robust stability of discrete-time systems, *Journal of Circuits, Systems and Computers*, 9(1 & 2):37-50, 1999.

[87] K. A. Ossman and E. D. Kamen, Adaptive regulation of MIMO linear discrete-time systems without requiring a persistent excitation, *IEEE Transactions on Automatic Control*, 32:397-404, 1987.

[88] R. V. Patel, A. J. Laub and P. M. Van Dooren, *Numerical Linear Algebra Techniques For Systems and Control*, IEEE Press, New York, 1994.

[89] F. M. Pait and S. A. Morse, A cyclic switching strategy for parameter adaptive control, *IEEE Transactions on Automatic Control*, 39: 1172-1183, 1994.

[90] J. W. Polderman, On the necessity of identifying the true parameter in adaptive lq control, *Systems & Control Letters*, 8:87-91, 1986.

[91] J. W. Polderman, *Adaptive Control & Identification: Conflict or Conflux?*, Ph.D. Thesis, University of Groningen, The Netherlands, 1987.

[92] J. W. Polderman, Avoiding the non-admissible region of the parameter space in indirect adaptive control algorithms, *Proc. of the 8th Intern. Conf. On Anal. and Optim. of Systems*, 822-829, 1988.

[93] J. W. Polderman and J. C. Willems, *Introduction to Mathematical Systems Theory, A Behavioral Approach*, Texts in Applied Mathematics 26, Springer, 1991.

[94] J. W. Polderman, A state space approach to the problem of adaptive pole assignment, *Mathematics of Control, Signals and Systems 2*, 2, Springer, 1989.

[95] M. Prandini, *Adaptive LQG Control: optimality analysis and robust controller design*, Ph.D. Thesis, University of Brescia, Italy, February 1998.

[96] M. Prandini and M. C. Campi, A new recursive identification algorithm for singularity free adaptive control, *Systems and Control Letters*, 34:177-183, 1998.

[97] M. Prandini M, M. C. Campi, Adaptive LQG control of input-output systems - A cost-biased approach. *SIAM J. Control and Optim*, **39**(5):1499-1519, 2001.

[98] L. Pronzato and E. Walter, Minimum-volume ellipsoids containing compact-sets-application to parameter bounding, *Automatica*, 30(11):1731-1739, 1994.

[99] W. Ren, The self-tuning of stochastic adaptive pole placement controllers, *Proc. 32nd Conf. on Decision and Control*, 1581-1586, 1993.

[100] A. Sabharwal and L. Potter, Set Estimation via Ellipsoidal Approximations: Theory and Algorithms, *Trans. on Signal Processing*, 45(12): 3107-3112, 1997.

[101] C. Samson, An adaptive LQ controller for non-minimum phase systems, *Intern. J. Control*, 1:1-28, 1982.

[102] J. Schoukens and R. Pintelon, Design of broadband excitation signals, *Perturbation signals for system identification*, Prentice Hall, Englewood Cliffs, NJ, 1993.

[103] J. H. van Schuppen, Tuning of guassian stochastic control systems, *IEEE Transactions on Automatic Control*, AC-39:2178-2190, 1994.

[104] F. C. Schweppe, Recursive state estimation: Unknown but bounded errors and systems inputs, *IEEE Transactions on Automatic Control*, 13(1):22-28, 1968.

[105] Special Issue on bounded-error estimation (Part II), *Int. Journal on Adaptive Control and Signal Processing*, 9(1): 1-132, 1995.

[106] J. F. Traub, G. Wasikowski and H. Wozniakowski, *Information-Based Complexity*, Academic Press, New York, 1988.

[107] S. M. Veres et al., *The Geometric Bounding Toolbox, Version 5.2.* MATLAB/SIMULINK Connections Catalog of MathWorks Inc., Licensed by The University of Birmingham, Web: www.eee.bham.ac.uk/gbt, 1993-98.

[108] L. Veselý, A Banach space in which all compact sets, but not all bounded sets, admit Chebishev centers, *Arch. Math.*, Birkhäuser Verlag, Basel, 79(6):499-506, 2002.

[109] A. Vicino and M. Milanese, Optimal inner bounds of feasible parameter set in linear estimation with bounded noise, *IEEE Trans. on Aut. Cont.*, 36(6): 759-763, 1991.

[110] A. Vicino and G. Zappa, Sequential approximation of parameter sets for identification with parametric and nonparametric uncertainty, *Proc. of the 32nd Conference on Decision and Control*, New York, 1993.

[111] S. Weiland, *Linear Matrix Inequalities in Control*, Lecture Notes, Disc Course, 2002.

[112] C. Wen and D. J. Hill, Global boundedness of discrete-time adaptive control using estimator projection, *Automatica*, 28:1143-1157, 1992.

[113] M. Wicks and R. A. DeCarlo, Computing the distance to an uncontrollable system, *IEEE Transactions on Automatic Control*, 36(1):39-49, 1991.

[114] Z. Zahreddine, On the Γ-stability of systems of differential equations in the Routh-Hurwitz and the Schur-Cohn cases, *Bulletin Belgian Math. Society Simon Stevin 3*: 363-368, 1996.

[115] O. Yaniv, P.-O. Gutman, L. Neumann, An algorithm for the adaptation of a robust controller to reduced plant uncertainty, *Automatica*, 4:709-720, 1990.

Summary

This thesis addresses a long-standing problem in adaptive control, that of the control system stability in the transient phase. Classically, adaptive control methods are based on the Certainty Equivalence Principle, according to the following ideas. At each iteration of the design, a model of the true system to be controlled is estimated using an identification procedure. Based on this model, a controller is designed to be applied to the real plant, as if there were no modeling error. As long as control performance is not satisfactory, the previous steps are repeated. However, when using such classical certainty equivalence principle based strategies, three problems are of concern. Firstly, because of model uncertainty there is by no means any guarantee that the time-frozen model-based controller will stabilize the true system. This may cause highly undesired transients in the control system behavior. More importantly, there is no way to check a priori if the model is controllable. Unfortunately, if controllability is not attained, no controller can be based on the model, implying a complete paralysis of the adaptive control scheme. Secondly, even in the case where at any frozen time the model is controllable and the controller based on this model stabilizes the real plant, if model time-variations are too fast, then asymptotic stability of the adaptive scheme may be destroyed.

To start with, the concept of *strong robustness*, fundamental in our work, is defined. A set of systems is said to be strongly robust with respect to a given control objective if it meets the following property: for any sequence of systems in this set, the time-varying controller based on this sequence of systems stabilizes any other fixed system in the set. In our adaptive control context, if we assume the model set to be strongly robust, then wherever the model is updated within this set and irrespective of how fast adaptation goes, the corresponding time-varying controller exists and stabilizes the true unknown plant. Hence, controllability of the model and stability of the time-varying closed-loop system are guaranteed over time, contrary to classical adaptive control approaches.

Following this idea, the main goal in this thesis is to design an adaptive control procedure exploiting the concept of strong robustness. To achieve this aim, our approach is threefold. As a first step, strong robustness is studied as a mathematical object (Chapter 3). In particular, attention is paid to the geometrical properties of strongly robust sets of systems for given control objectives, mainly pole placement design and linear quadratic control. The systems under consideration are linear and time-invariant SISO systems in discrete-time description, with an unknown-but-bounded modeling error with known upper and lower bounds and with a known order. A fundamental result is the existence of non-trivial strongly robust neighborhoods around any system in the considered class of systems. Then, sufficiency tests for the characterization of strongly robust sets of systems have been expressed by means of various

control theory tools such as linear matrix inequalities and a Kharitonov-like test.
The second step in our approach (Chapter 4) is to relate the concept of strong robustness to identification for adaptive control. In this respect, attention is paid to the following question: in the perspective of identification for adaptive control, how to obtain a strongly robust set of models? To solve this identification issue, we design and implement an open-loop identification input design ensuring that the uncertainty set becomes strongly robust in finite time.
The third and last step in our approach is to revisit classical adaptive control exploiting the notion of strong robustness, so as to yield the so-called *strongly robust adaptive control* (Chapter 5). At each time of the design, instead of blindly using the model to achieve the control design as it is commonly done in classical adaptive control approaches, one first checks whether the set of all model candidates is strongly robust. Once this condition is met, which is guaranteed to happen in finite time, one then proceeds to control using a classical certainty equivalence type of strategy. The developed adaptive control scheme hence splits in two phases. In the first phase, focus is mainly put on off-line identification of a strongly robust model set. At each time instant, a criterion tells whether strong robustness is achieved or not. When this criterion is satisfied, the adaptive control switches to the second phase, the control phase, where effort is shifted to control according to a certainty equivalence strategy. Proceeding in this way, one secures asymptotic stability of the closed-loop system, whilst ensuring that initial uncertainty will not yield undesired transients.
The strong robustness-based adaptive control method is presented in a general framework. Finally, particular attention is paid to the case of strongly robust adaptive pole placement design for which a detailed analysis and some implementation aspects have been proposed.

Samenvatting

Dit proefschrift behandelt een bekend probleem in de adaptieve regeltechniek, te weten de stabiliteit van het regelsysteem in de initiële fase. Traditioneel zijn adaptieve regelmethoden gebaseerd op het zekerheids equivalentie principe. Tijdens elke iteratie wordt een model van het te regelen systeem geschat met behulp van een identificatieprocedure. Gebaseerd op dit model wordt een regelaar ontworpen die op het werkelijke systeem wordt toegepast alsof er geen modelleerfout is. Zolang de prestaties van het regelsysteem niet bevredigend zijn, worden de voorgaande stappen herhaald. Met algoritmes gebaseerd op het zekerheids equivalentie principe zijn drie problemen te onderscheiden. Ten eerste is er vanwege de modelonzekerheid geen enkele garantie dat een regelaar gebaseerd op het model het werkelijke systeem zal stabiliseren. Dit kan zeer ongewenste overgangsverschijnselen veroorzaken. Ten tweede is er geen manier is om van tevoren te garanderen dat het model regelbaar is. Helaas kan er geen regelaar op het model gebaseerd worden als het model niet regelbaar is. Ten derde kan, zelfs in het geval dat op elk tijdstip het model regelbaar is en de regelaar gebaseerd op dit model het werkelijke systeem stabiliseert, de asymptotisch stabiliteit van het adaptieve schema teniet worden gedaan als tijdvariaties in het model te snel zijn.

In dit proefschrift wordt eerst het concept *sterke robuustheid*, dat fundamenteel is in ons werk, gedefinieerd. Een verzameling systemen wordt met betrekking tot een gegeven regeldoel sterk robuust genoemd als het voldoet aan de volgende eigenschap: voor elke rij systemen in deze verzameling stabiliseert de tijdvariërende regelaar gebaseerd op deze rij systemen elk vast systeem in de verzameling. In onze context van adaptieve regeltechniek is het zo dat als de verzameling modellen sterk robuust is, dan bestaat de bijbehorende tijdvariërende regelaar en deze stabiliseert het onbekende werkelijke systeem, onafhankelijk van hoe het model wordt aangepast binnen deze verzameling en ongeacht de snelheid van de aanpassing. Hierdoor worden regelbaarheid van het model en stabiliteit van het tijdvariërende gesloten-lus systeem gegarandeerd. Eigenschappen die klassieke adaptieve regelmethoden zonder verder modificaties vaak niet hebben.

Het uitwerken van bovenstaand idee vormt het hoofddoel van het onderzoek in dit proefschrift: het ontwerpen van een adaptieve regelprocedure die gebruikt maakt van het concept van sterke robuustheid. Om dit doel te bereiken is de structuur van het proefschrift drieledig. Als een eerste stap wordt sterke robuustheid bestudeerd als een wiskundig concept (Hoofdstuk 3). In het bijzonder wordt aandacht besteed aan de geometrische eigenschappen van verzamelingen van systemen die sterk robuust zijn. De gebruikte regeldoelen zijn voornamelijk poolplaatsing en lineair kwadratisch regelen. De bestudeerde systemen zijn lineaire en tijdinvariante SISO-systemen in discrete tijd, met een onbekende-maar-begrensde modelleerfout met bekende boven- en ondergrenzen en van een bekende orde. Een fundamenteel resultaat is het bestaan van niet-triviale sterk robuust omgevingen rondom elk systeem in de beschouwde systeemklasse. Vervolgens worden voldoende voorwaarden afgeleid voor de karakterisering van sterk robuuste verzamelingen van systemen. De gebruikte technieken zijn o.a. lineaire matrix ongelijkheden en een Kharitonov achtige test.

De tweede stap in onze aanpak (Hoofdstuk 4) is het combineren van het concept van sterke robuustheid met identificatie. Met betrekking daartoe wordt aandacht geschonken aan de volgende vraag: hoe kan een sterk robuuste verzameling modellen verkregen worden in het perspectief van identificatie ten behoeve van regelen? Om dit identificatievraagstuk op te lossen, wordt een open-lus identificatie-ingang ontworpen en geïmplementeerd, die garandeert dat de onzekerheidsverzameling sterk robuust wordt binnen een eindig aantal stappen.

De derde en laatste stap in onze aanpak is het modificeren van de klassieke adaptieve regeltechniek gebruikmakende van het concept sterke robuustheid. Dit leidt tot wat genoend zou kunnen worden sterk robuuste adapieve regelsystemen (Hoofdstuk 5). In plaats van blind het model te gebruiken om het regelontwerp te bereiken, zoals gebruikelijk is in klassieke adaptieve regelmethodes, wordt nu op elk tijdstip van het ontwerp eerst gecontroleerd of de verzameling van alle modelkandidaten sterk robuust is. Als eenmaal aan deze voorwaarde is voldaan, wat gegarandeerd binnen eindige tijd gebeurt, gaat men verder met het regelen volgens een klassieke zekerheids equivalente strategie. Het ontwikkelde adaptieve regelschema kan dus opgesplitst worden in twee fasen. In de eerste fase ligt de nadruk grotendeels op open lus identificatie van een sterk robuuste modelverzameling. Op elk tijdstip laat een criterium zien of sterke robuustheid wel of niet bereikt is. Als aan dit criterium voldaan wordt, schakelt de adaptieve regeling over naar de tweede fase, de regelfase, waarin de nadruk wordt verschoven naar het regelen van het systeem. Door op deze manier verder te gaan, behoudt men asymptotische stabiliteit van het gesloten-lus systeem, terwijl men er tegelijkertijd van verzekerd is dat initiële onzekerheid geen ongewenste overgangsverschijnselen zal opleveren. De op sterke robuustheid gebaseerde adaptieve regelmethode wordt gepresenteerd in een algemeen kader. Tenslotte wordt in het bijzonder aandacht besteed aan het geval van sterk robuust poolplaatsingsontwerp, waarvan een gedetailleerde analyse is gemaakt en waarvoor enkele implementatie-aspekten worden bekeken.

Résumé

Cette dissertation traite du problème souvent rencontré en commande adaptative de procédés qu'est celui de la stabilité du système commandé en phase transitoire. Les méthodes classiques de synthèse de commandes adaptatives s'inspirent du Principe de l'Equivalence Certaine, basé sur les idées suivantes. A chaque itération de la synthèse du contrôleur, un modèle du système à commander est estimé au moyen d'une procédure d'identification. A partir de ce modèle, un contrôleur est synthétisé puis appliqué au système à commander, et cela sans tenir compte des erreurs de modélisation. Tant que les performances du système contrôlé ainsi constitué ne sont pas jugées satisfaisantes, l'algorithme précédemment décrit est re-itéré. Une telle stratégie de synthèse de commande s'accompagne cependant de trois problèmes. Tout d'abord, en raison des erreurs de modélisation, le concepteur n'a en aucun cas la guarantie que le contrôleur appliqué stabilise à tout instant le système à commander, de sorte que peuvent se produire des transitoires d'amplitude déraisonnable dans le comportement entrée-sortie du système de commande. Le second problème vient du fait qu'il n'y a pas moyen de vérifier a priori si le modèle est controlable. Si malheureusement il ne l'est pas, auncun contrôleur ne peut être synthétisé à partir de ce modèle, ce qui entraine une paralysie totale de l'algorithme de contrôle. De plus, même dans le cas favorable où le modèle est controlable à chaque instant et donc possède une loi de commande qui stabilise le système à commander, la stabilité du système peut être perdue si le modèle varie trop rapidement.

Dans un premier temps, nous défininissons la notion de *robustesse forte* (strong robustness) qui joue un rôle fondamental dans notre travail. Un ensemble de systèmes S est dit *fortement robuste* par rapport à un objectif de control fixé si il possède la propriété suivante: étant donnée une famille de sytèmes appartenant à cet ensemble S, le contrôleur variable généré par cette séquence de systèmes stabilise tout autre système élément de cet ensemble S. Dans le contexte de contrôle adaptatif qui est le nôtre, sous l'hypothèse que l'ensemble de modèles que nous considérons est fortement robuste, alors nous avons le résultat suivant: où que soit choisi le modèle dans cet ensemble de systèmes, et quelle que soit la vitesse avec laquelle ce modèle est remplacé, le contrôleur variable qui lui correspond est défini à tout moment et stabilise le système à commander. Par conséquent, la controlabilité du modèle et la stabilité asymptotique du système de contrôle en boucle fermée sont guarantis, contrairement à ce que nous pourrions obtenir en utilisant les méthodes classiques de commande adaptative.

Le but principal de cette thèse est de développer une procédure de contrôle adaptatif exploitant le concept de robustesse forte. Pour atteindre cet objectif, notre étude est menée en trois temps. Tout d'abord, la notion de robustesse forte est étudiée en tant que concept mathématique (Chapitre 3). En particulier, nous nous intéressons aux propriétés géométriques des ensembles de systèmes fortement robustes, principalement dans un cadre de synthèse

de lois de commande avec placement de pôles ou de la méthode de commande linéaire quadratique. Les systèmes que nous considérons sont linéaires, invariants dans le temps, mono-entrée mono-sortie (SISO), discrets et d'ordre connu. L'erreur de modélisation est quand à elle supposée inconnue mais bornée, et possèdant une borne supérieure et une borne inférieure connues. Le résultat suivant est établi: autour de tout système dans la classe de systèmes étudiés, il existe un voisinage de systèmes qui possède la propriété de robustesse forte. Ensuite, des tests charactérisant les ensembles de systèmes fortement robustes sont exprimés au moyen d'outils empruntés à la théorie de la commande de systèmes, tels que des inégalités matricielles linéaires et le test de Kharitonov.

Ensuite, dans la deuxième partie de notre approche (Chapitre 4), nous relions les concepts de robustesse forte et d' identification pour le contrôle adaptatif. Pour ce faire, nous nous posons la question suivante: dans une perspective d'identification en vue d'une synthèse de lois de commande, quel signal d'entrée utiliser afin d'obtenir des ensembles de modèles fortement robustes? En réponse à cette question, nous synthétisons et implémentons un signal d'entrée d'identification en boucle ouverte assurant que l'ensemble des modèles identifiés devient fortement robuste en temps fini.

Enfin, dans la troisième partie de notre étude nous reconsidérons la stratégie classique de synthèse de lois de commandes adaptatives à la lumière de notre précédente étude de la notion de robustess forte. Ceci nous conduit à un algorithme de synthèse de systèmes de contrôle adaptatif incluant la propriété de robustesse forte (Chapitre 5). A chaque itération, au lieu d'utiliser aveuglément le modèle pour la synthèse du contrôleur comme le font les méthodes usuelles de synthèse de systèmes de de contrôle adaptatif, nous vérifions tout d'abord si l'ensemble des modèles identifiés est fortement robuste. Une fois la condition de robustesse forte remplie, ce qui est guaranti en temps fini, alors une approche basée sur le principe de l'équivalence certaine est alors employée. L'algorithme de synthèse de lois de commandes adaptatives faisant l'objet de cette thèse s'articule donc autour de deux phases. Tout au long de la première phase, l'effort est surtout mis dans l'identification "off-line" d'un ensemble de modèles fortement robuste. A chaque instant, un test est utilisé pour savoir si il y a robustesse forte ou pas. Lorsque ce test est positif, le système adaptatif entre dans la seconde phase, ou l'effort est cette fois mis au service du contrôle selon le principe d'équivalence certaine. De cette facon, nous assurons la stabilité asymptotique du système commandé, tout en garantissant que l'incertitude sur le système à commander ne donne pas naissance à des transitoires de trop grande amplitude.

Notre méthode de synthèse de systèmes de contrôle adaptatif basée sur la notion de robustesse forte est présentée dans un contexte aussi général que possible. Ensuite, le cas de placement de pôles fait l'objet d'une étude plus détaillée et nous permet de simuler certains de nos résultats.

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Maria Cadic in Boselli

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Maria Cadic was born on December 21, 1975 in Toulouse, France. After graduating from high school at the *Lycée Pape Clément*, Pessac, France, in 1994, she began to study Mathematics, Physics and Chemistry in the *Classes Préparatoires aux Grandes Ecoles*, at the *Lycée Michel Montaigne*, in Bordeaux, France. In the meantime, she obtained her university diploma *Diplôme d'Etudes Universitaires Générales* in Physics and Chemistry at the *Université des Sciences de Bordeaux I* in June 1995. In 1997, she joined the Engineering School *Institut des Sciences de la Matière et du Rayonnement*, Caen, France, from which she obtained her Engineering Diploma in Electronic Systems in July 1999. During 1998/1999, she also completed a Master of Engineering in Electronic Systems at the *School of Electronic Engineering*, at *Dublin City University*, Ireland, for which she graduated in October 1999 with First Class Honour. Her master's thesis, supervised by Dr. Tony Holohan, was entitled "System Identification for Control".

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